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MATHEMATICAL DUEL '08

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1. vydání

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Preface

The 16th international mathematics competition "Duel" took place in Přerov at the beginning of April 2008.

This competition started in 1993, two teams of mathematically gifted high school students (GMK Bílovec and I LO Chorzów, Poland) participated in its first run in Bílovec. The competition included only two high school categories. The team of BRG Kepler in Graz (Austria) has joined since the 5th year of the competition (1997) and the competition has changed into three-lateral. Its organization changes cyclically (Bílovec, Chorzów, Graz). The name of the competition stayed the same with a bit change meaning of this word (duel = fight) only. Organizing schools of given year invited some guest teams usually. The structure of the competition has not changed since its 3rd year. Students of the last two years of eight-year grammar school solve A category, fifth and sixth graders solve B category, and contestants of lower grammar school of the same school solve the lowest C category. The competition takes place in two parts in all three categories—an individual competition and a team competition. Each school is represented in individual competition by four contestants. Special problems are prepared for each category. The text of given problems is introduced to contestants in English, but the solutions and the results can be written in their mother tongue.

Six schools took part in the 16th run of the competition, besides the three traditional schools, there were also three schools from the region of Olomouc (Jakub Škoda Gymnasium in Přerov—as the organizer school, Slavic Gymnasium in Olomouc and Gymnasium Olomouc–Hejčín). The total number of the contestants in the 16th year of Mathematical Duel was 72 then. The competition runs in this extent mainly due the supporting by a grant of MŠMT ČR, NPV II, No. 2E06029, STM Morava.

Authors

Problems

Category A (Individual Competition)

A–I–1

Show that
$$n = \frac{2008^3 + 2007^3 + 3 \cdot 2008 \cdot 2007 - 1}{2009^2 + 2008^2 + 1}$$
is an integer and determine its value.

A–I–2

The orthocenter H of an acute-angled triangle ABC is reflected on the sides a, b and c yielding points A_1, B_1 and C_1, respectively. We are given that

$$|\angle C_1AB_1| = |\angle CA_1B|,\ |\angle A_1BC_1| = |\angle AB_1C| \text{ and } |\angle B_1CA_1| = |\angle BC_1A|$$

hold. Prove that ABC must be the equilateral triangle.

A–I–3

Let a, b, c be arbitrary positive real numbers such that $abc = 1$. Prove that the inequality

$$\frac{a}{ab+1} + \frac{b}{bc+1} + \frac{c}{ca+1} \geq \frac{3}{2}$$

holds. When does the equality hold?

A–I–4

Let $ABCD$ be a tetrahedron with all three mutually perpendicular edges at its vertex D. Further let us denote S the center of its circumscribed sphere. Prove that the centroid T of its face ABC lies on the line DS.

Category A (Team Competition)

A–T–1

Determine all triples (x, y, z) of positive integers such that the equality

$$3 + x + y + z = xyz$$

holds.

A–T–2

Let D be a point on the side BC of the given triangle ABC such that

$$|AB| + |BD| = |AC| + |CD|$$

holds. The line segment AD intersects the incircle of the triangle ABC at X and Y with X closer to A. Let E be the point of tangency of the incircle of the triangle ABC with its side BC. Show that

a) The line EY is perpendicular to AD.
b) The equality $|XD| = 2\,|IM|$ holds (I denotes the incentre of the triangle ABC and M is the midpoint of the line segment BC).

A–T–3

Let $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all integers and real numbers, respectively. Determine all functions $f : \mathbb{Z} \to \mathbb{R}$ such that the condition

$$f(3x - y) \cdot f(y) = 3f(x)$$

is fulfilled for all integers x and y.

Category B (Individual Competition)

B–I–1

Show that

$$m = \frac{2008^4 + 2008^2 + 1}{2008^2 + 2008 + 1}$$

is an integer and determine its value.

B–I–2

Determine all positive integers n for which there exist positive integers x and y satisfying

$$x + y = n^2,$$
$$10x + y = n^3.$$

B–I–3

Let $ABCD$ be a trapezoid ($AB \parallel CD$) of unit area, with $|AB| = 2\,|CD|$. Further let K and L be midpoints of its sides BC and CD respectively. Determine the area of the triangle AKL.

B–I–4

The first (lead) digit of a positive integer N is 3. If we write the same integer without the lead digit 3, we obtain an integer M.

a) Determine all such integers N, for which $N = A \cdot M$ holds with $A = 25$.
b) Determine two further positive integers $A \neq 25$ for which such integers N and M exist.
c) Prove that no such integers N and M exist for $A = 32$.

Category B (Team Competition)

B–T–1

Let $ABCD$ be a convex quadrilateral in the plane. Further let K, L, M and N be midpoints of its sides AB, BC, CD and DA respectively.

a) Prove that there exists a triangle with sides of lengths KL, KM and KN.
b) Determine its area Q depending on the area P of the quadrilateral $ABCD$.

B–T–2

The average (arithmetic mean) of 10 positive integers is 2 008.

a) What is the largest possible number among them and what is the smallest?
b) If 304 is one of the numbers, what is largest possible number among them and what is the smallest?
c) If we know that all 10 positive integers are different, what is the largest and smallest possible value for the largest of the 10 positive numbers?

B–T–3

Of the triples $(4, 6, 8)$, $(4, 8, 9)$ and $(5, 12, 13)$ only one can be interpreted as giving the lengths of the three altitudes of a triangle ABC.

a) State which of these triples can give the altitudes of a triangle.
b) Give a compass and staight-edge (euclidean) construction for the triangle whose altitudes are given by the triple (in cm), and prove that the construction is complete.

Category C (Individual Competition)

C–I–1

Determine the number of all triangles with integer-length sides, such that two of sides are lengths of m and n ($1 \le m \le n$). Solve this problem for the specific values $m = 6$ and $n = 9$ and then for general values.

C–I–2

We are given two parallels p and q in the plane. Let us consider the set A of 13 different points such that 7 of them lie on p and the other 6 lie on q.

a) How many segment lines form the points of the set A (as end-points of segment lines)?
b) How many triangles form the points of the set A (as vertices of triangles)?

C–I–3

Determine the smallest positive integers x and y such that the equality

$$12x = 25y^2$$

holds.

C–I–4

Prove that all three medians of the given triangle cut this triangle into six smaller triangles of the same area.

Category C (Team Competition)

C–T–1

The line segment XY with $|XY| = 2$ is common (main) diagonal of a regular hexagon and a square. Determine the area of the section common to both figures.

C–T–2

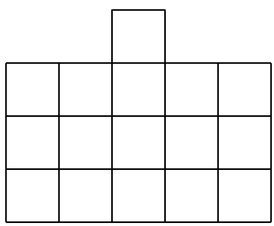
Let a, b, c be real numbers. Prove that

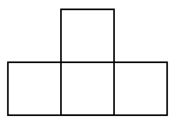
$$V = 4(a^2 + b^2 + c^2) - [(a+b)^2 + (b+c)^2 + (c+a)^2]$$

is always non-negative real valued and determine all values of a, b, c for which $V = 0$ holds.

C–T–3

We are given a board composed of 16 unit squares as shown. We wish to colour some of the cells green in such a way that, no matter where we place the T-shaped tetromino on the board (with each square of of the tetromino covering exactly one square on the board), at least one square of the tetromino will be on a green cell. Determine the smallest possible number of cells we must colour green and prove that this is the smallest number.





Solutions

Category A (Individual Competition)

A–I–1

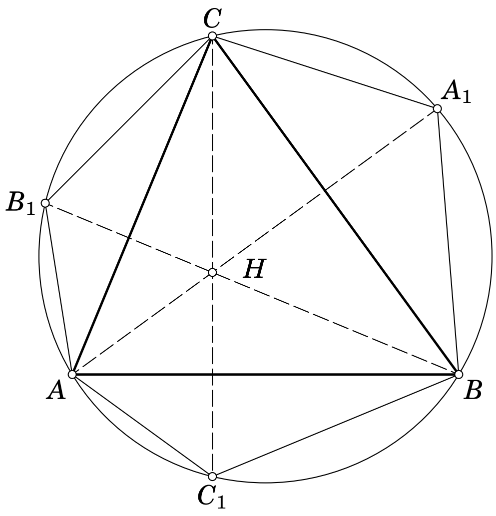
Substituting $x = 2008$, we obtain

$$n = \frac{x^3 + (x-1)^3 + 3x(x-1) - 1}{(x+1)^2 + x^2 + 1}$$
$$= \frac{2x^3 - 2}{2x^2 + 2x + 2} = x - 1$$

and therefore $n = 2008 - 1 = 2007$.

A–I–2

Naming $|\angle CAB| = \alpha$, $|\angle ABC| = \beta$ and $|\angle BCA| = \gamma$, we first note that $|\angle BA_1C| = |\angle BHC| = 180^\circ - |\angle HBC| - |\angle HCB| = 180^\circ - (90^\circ - \gamma) - (90^\circ - \beta) = \beta + \gamma$ holds. Since $\alpha = 180^\circ - (\beta + \gamma)$, we see that A_1 must lie on the circumcircle of ABC. (This fact is well known.) Similarly, B_1 and C_1 must also lie on the circumcircle of ABC, and we therefore have $|\angle CAB_1| = |\angle CBB_1| = 90^\circ - \gamma$ and $|\angle BAC_1| = |\angle BCC_1| = 90^\circ - \beta$. It therefore follows that $|\angle C_1AB_1| = \alpha + 90^\circ - \gamma + 90^\circ - \beta = 2\alpha$. Since $|\angle C_1AB_1| = |\angle CA_1B|$, we therefore have $2\alpha = \beta + \gamma = 180^\circ - \alpha$, and therefore $\alpha = 60^\circ$.



Since the same holds for β and γ, the triangle ABC must be equilateral.

A–I–3

Since $abc = 1$ there exist positive real numbers x, y, z such that $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$. Then

$$\frac{a}{ab+1} + \frac{b}{bc+1} + \frac{c}{ca+1} = \frac{\frac{x}{y}}{\frac{x}{z}+1} + \frac{\frac{y}{z}}{\frac{y}{x}+1} + \frac{\frac{z}{x}}{\frac{z}{y}+1} =$$
$$= \frac{zx}{xy+yz} + \frac{xy}{zx+yz} + \frac{yz}{xy+zx} \geq \frac{3}{2}$$

where the inequality follows from so-called *Nesbitt's* inequality which is well-known in the following form: For arbitrary positive real numbers r, s, t

$$\frac{r}{s+t} + \frac{s}{t+r} + \frac{t}{r+s} \geq \frac{3}{2}$$

holds, with equality in the case $r = s = t$.

Therefore, equality holds (in the given problem) iff

$$a = b = c = 1.$$

A–I–4

We can use a Cartesian coordinate system $Oxyz$ such that $D = O$ and the edges DA, DB and DC lie on axes x, y and z, respectively. Let a, b, c be the lengths of edges DA, DB and DC, respectively. The center S of the circumsphere of the tetrahedon $ABCD$ has coordinates $[\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c]$. For each point $P[x, y, z]$ of the ray DS the following two conditions are fullfiled

$$xc = az \qquad \text{and} \qquad yc = bz.$$

Therefore the ray DS passes through the point $T[\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c]$ which is the centroid of the face ABC of given tehrahedron $ABCD$ and the proof is finished.

Category A (Team Competition)

A–T–1

By symmetry, we can assume without loss of generality, that $1 \le x \le$ $\le y \le z$ holds.

If $x = 1$, the equation can be expressed as $y + 4 = z(y - 1)$ (which is incorrect for $y = 1$), and we therefore have $z = \frac{y+4}{y-1} = 1 + \frac{5}{y-1}$ for $y > 1$. For $y = 2$, we get $z = 6$, and since $3 + 1 + 2 + 6 = 12$ is a true statement, we see that all permutations of $(1; 2; 6)$ are solutions of the equation. We shall now show that there are no others.

Staying with the case $x = 1$, we note that $z(y) = 1 + \frac{5}{y-1}$ is decreasing. $z(3) = \frac{7}{2}$ is not an integer, and $z(4) = \frac{8}{3}$ is smaller than $y = 4$. Since $z(y)$ is decreasing, the existence of any further solutions would contradict $y \le z$.

For $x = 2$, we obtain

$$z(y) = \frac{y + 5}{2y - 1} = \frac{1}{2} + \frac{7}{2y - 1}.$$

We see that $z(2) = \frac{7}{3}$ is not an integer, and since $z(3) = \frac{8}{5} < 3$, any further solutions in this case would once more contradict $y \le z$

For any given $x \ge 3$, we similarly obtain

$$z(y) = \frac{y + (x + 3)}{xy - 1} = \frac{1}{x} + \frac{x + 3 + \frac{1}{x}}{xy - 1},$$

which is once again decreasing. For $y = x$, we have

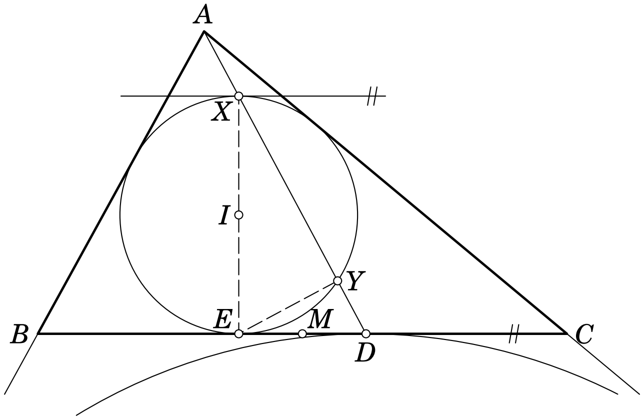
$$z(y) = \frac{2x + 3}{x^2 - 1} = \frac{2}{x + 1} + \frac{1}{x^2 - 1},$$

which is also decreasing as a function in x, and since $z(3) = \frac{9}{8}$ is already smaller than 3, there can certainly be no further solutions of the equation in positive integers.

A–T–2

a) The homothety mapping the excircle at the side BC of triangle ABC (tangent to BC at the point D) to the incircle of the same triangle maps the point D to X. Therefore a tangent at X is parallel to BC. XE is therefore a diameter of the incircle and thus XYE is a right angle.

b) By easy computation we obtain $|BE| = |DC| = s - b$, where $2s = a + b + c$. So M is the midpoint of the segment ED again. Thus, by the midpoint theorem $|XD| = 2\,|IM|$ holds.



A–T–3

Let f be any such function. Setting $x = y = 0$ we get $f(0) \in \{0, 3\}$.

If $f(0) = 0$, we set $y = 0$ and get $f(x) = 0$, so $f(x) \equiv 0$ is the first solution.

Let $f(0) = 3$. We set $x = 0$ and get $f(y)f(-y) = 9$, particularly $f(y) \neq 0$ for all integers y. When we set $x = y \neq 0$ we get $f(2x) = 3$. Thus $f(x) = 3$ for all even x. Setting $y = 4x$ gives $f(-y) = f(y)$. Since $f(y)f(-y) = 9$, it follows that $f(y) \in \{-3, 3\}$ and f is an even function.

Setting $x = 1$, $y = 2t + 1$ (t is an integer) we get

$$f(2 - 2t) \cdot f(2t + 1) = 3f(1)$$

from which it follows that $f(2t + 1) = f(1)$. Checking we see that $f(x) \equiv 3$ is the second solution and $f(x) = \begin{cases} 3 & x \text{ even} \\ -3 & x \text{ odd} \end{cases}$ is the third solution.

Category B (Individual Competition)

B–I–1

Noting that $(x^4 + x^2 + 1) : (x^2 + x + 1) = x^2 - x + 1$, we only need to substitute $x = 2008$ to obtain $M = 4030057$.

B–I–2

From the first equation $y = n^2 - x$ and substituting to the second equation we obtain $9x = n^2(n-1)$. The left side is divisible by 3.

Let us consider two possible cases:

- ▷ n is divisible by 3, i.e. $n = 3k$ for $k \in \mathbb{N}$. Then

$$9x = 9k^2(3k-1), \qquad x = k^2(3k-1) = 3k^3 - k^2$$

and

$$y = (3k)^2 - (3k^3 - k^2) = 10k^2 - 3k^3.$$

Because $y \geq 1$ then $10k^2 - 3k^3 \geq 1$, $k^2(10-3k) \geq 1$ and it follows that $10 - 3k \geq 1$, $k \leq 3$.

Then $k \in \{1, 2, 3\}$ and $n \in \{3, 6, 9\}$. For this n the x and y are positive integer numbers.

- ▷ $n - 1$ is divisible by 3.

Then n is not divisible by 3 and $n - 1$ is divisible by 9. We obtain $n - 1 = 9m$ for some nonnegative integer m. Then $n = 9m + 1$,

$$9x = (9m+1)^2 \cdot 9m, \quad x = m(9m+1)^2, \quad y = n^2 - x,$$
$$y = (9m+1)^2 - m(9m+1)^2 = (9m+1)^2(1-m).$$

Because $y \geq 1$, then $(9m+1)^2(1-m) \geq 1$ and it follows $1 - m \geq 1$, $m \leq 0$. It is impossible for $m \in \mathbb{N}$.

The solution of the system has in positive integers is therefore possible for $n = 3, 6, 9$.

Another solution. Analogically to the above $x = \frac{1}{9}n^2(n-1)$. Let us consider three possible cases:

- ▷ $n = 3k$, $k \in \mathbb{Z}$. Then $x = \frac{1}{9} \cdot 9k^2(3k-1) = k^2(3k-1) \in \mathbb{Z}$.

▷ $n = 3k + 1$, $k \in \mathbb{Z}$. Then

$$x = \frac{1}{9}(9k^2+6k+1)(3k-1) = \frac{1}{9}(27k^3+9k^2-3k-1) = 3k^3+k^2-\frac{3k+1}{9} \notin \mathbb{Z}.$$

▷ $n = 3k + 2$, $k \in \mathbb{Z}$. Then

$$x = \frac{1}{9}(9k^2 + 12k + 4)(3k - 1) = \frac{1}{9}(27k^3 + 9k^2 - 4) = 3k^3 + 3k^2 - \frac{4}{9} \notin \mathbb{Z}.$$

Then there must be $n = 3k$ for $k \in \mathbb{N}$. From the first solution $x = k^2(3k - 1)$. Because $y \geq 1$ and $y = 10k^3 - 3k^2 = k^2(10k - 3)$, then $10 - 3k \geq 1$ and we obtain $k \leq 3$.

Then $k \in \{1, 2, 3\}$ and $n \in \{3, 6, 9\}$. For the parameters $n \in \{3, 6, 9\}$ the solutions x and y are positive integer numbers.

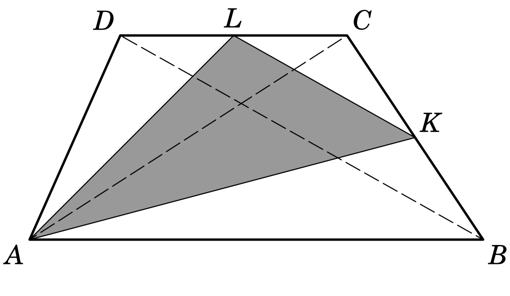
Another solution. From the first solution we have $x = \frac{1}{9}n^2(n-1)$, $x \in \mathbb{N}$, then $n - 1 \geq 1$ and $n \geq 2$. Therefore $y = \frac{n^2}{9}(10 - n)$. From $y \geq 1$ it follows $10 - n \geq 1$, i.e. $n \leq 9$.

It may be verified that for an integer n, $2 \leq n \leq 9$ only for $n \in \{3, 6, 9\}$ x and y are positive integer numbers.

B–I–3

From the picture we can see that the areas of the triangles *ABC*, *BCD* and *CDA* are $\frac{2}{3}$, $\frac{1}{3}$ and $\frac{1}{3}$, respectively. Thus, for the area S of the triangle *AKL* we have

$$S = S_{ABCD} - S_{ABK} - S_{ADL} - S_{CLK} = 1 - \frac{1}{2} \cdot \frac{2}{3} - \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{4} \cdot \frac{1}{2} = \frac{5}{12}.$$



B–I–4

The given condition can be written as $n = 3 \cdot 10^k + m = am$.

a) For $a = 25$, we have

$$3 \cdot 10^k + m = 25 \cdot m \iff 3 \cdot 10^k = 24m \iff 10^k = 8m \iff 125 \cdot 10^{k-3} = m,$$

and the required numbers are given by

$$n = 3125,\ 31250,\ 312500,\ \ldots$$

b) Two examples are $a = 31$, for which we get

$$n = 31,\ 310,\ 3100,\ \ldots$$

and $a = 13$, for which we get $10^k = 4m \iff 25 \cdot 10^{k-2} = m$, and therefore

$$n = 325,\ 3250,\ 32500,\ \ldots$$

c) $a = 32$ yields $3 \cdot 10^k = 31 \cdot a$, which is impossible, since 31 cannot be a divisor of $3 \cdot 10^k$.

Category B (Team Competition)

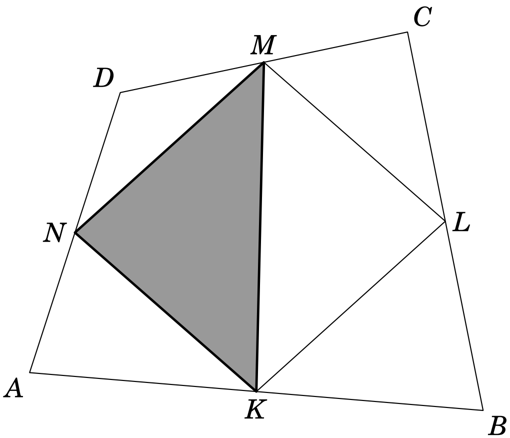
B–T–1

First of all, we can see that the quadrilateral *KLMN* is a parallelogram (the so-called *Varignon's* parallelogram).

a) Since the sides *KL* and *MN* of the parallelogram *KLMN* are of equal length, the triangle *KMN* exists whose sides are congruent with line segments *KL*, *KM* and *KM*. This concludes our proof.

b) For the area Q of the parallelogram *KLMN* we have

$$Q = P - \frac{1}{4}P - \frac{1}{4}P = \frac{1}{2}P.$$



B–T–2

If the average of ten numbers is 2008, their sum is $10 \cdot 2008 = 20080$.

a) Since all numbers are positive integers, the smallest value for any of them is 1. If nine of the numbers are as small as possible, i.e. equal to 1, the tenth can be $20080 - 9 \cdot 1 = 20071$, and this is the largest possible number.

b) If one of the numbers is 304, the sum of the other 9 is $20080 - 304 = 19776$. Choosing 8 ones (once more the smallest possible), the largest possible number is $19776 - 8 \cdot 1 = 19768$.

c) In order to obtain the largest possible number, we must choose the other 9 as small as possible. These are $1, 2, \ldots, 9$, and the sum of these numbers is 45. The largest possible number in this case is therefore $20080 - 45 = 20035$.

In order for the largest number to as small as possible, the numbers must vary as little as possible from 2008. Consider the numbers

$$2003, 2004, 2005, 2006, 2007, 2009, 2010, 2011, 2012, 2013.$$

Their sum is 20080, and their average therefore 2008. If the largest number (2013) is made smaller, some other number must be made larger by the same amount without resulting in some number being in the list twice. It is possible to reduce by 5 (to 2008), but then some other number must be raised by 5, yielding either a number already in the list or a number larger than 2012. If it is reduced by more than 10, some other number must be enlarged by more than 10, yielding a number larger than 2012. We see that the smallest possible value for the largest number in the list is 2013.

B–T–3

Since $ah_a = bh_b = ch_c = 2A$, we have $a = \frac{2A}{h_a}$, $b = \frac{2A}{h_b}$ and $c = \frac{2A}{h_c}$, and the triangle inequality therefore yields

$$a < b + c \quad \Rightarrow \quad \frac{1}{h_a} < \frac{1}{h_b} + \frac{1}{h_c}.$$

Since $\frac{1}{5} < \frac{1}{12} + \frac{1}{13}$ and $\frac{1}{4} < \frac{1}{8} + \frac{1}{9}$ are both incorrect, the triples $(4, 8, 9)$ and $(5, 12, 13)$ cannot denote the altitudes of triangles. The inequality does, however hold for the triple $(4, 6, 8)$.

Since the sides of a triangle with altitudes $(4, 6, 8)$ must be in the ratio $\frac{1}{4} : \frac{1}{6} : \frac{1}{8} = 6 : 4 : 3$, one possible construction is to construct any triangle whose sides are in this proportion and then expand or contract such that one of the altitudes assumes the appropriate size. (Parallel line to the side whose distance is equal to the altitude; then application of homothety from one of the vertices.)

Category C (Individual Competition)

C–I–1

In the general case observe that the length of the third side is greater than $n + m - 1$ and less then $n - m + 1$. Between $n + m - 1$ and $n - m + 1$ we have $(n + m - 1) - (n - m + 1) + 1 = 2m - 1$ positive integer numbers.

The number of considered triangles is therefore $2m - 1$.

In the particular case $m = 6$ and $n = 9$ there are 11 triangles.

C–I–2

a) For one point from A we may form 12 segments. For 13 points from A it is possible form $13 \cdot 12 = 156$ segments, but every segment was counted twice, we therefore only have 78 segments.

b) Observe that for 7 points from the line p it is possible to form $\frac{7 \cdot 6}{2} = 21$ different segments. If the third vertex of the triangle is on the line q, it is possible to form $21 \cdot 6 = 126$ triangles.

It is impossible to form a triangle if all three points lie on the line p or the line q.

Analogically for 6 points from the line q it is possible to form $\frac{6 \cdot 5}{2} = 15$ different segments. If the third vertex of the triangle is from the line p, then it is possible to form $15 \cdot 7 = 105$ triangles.

It is possible to form $126 + 105 = 231$ different triangles with vertices from the set A.

C–I–3

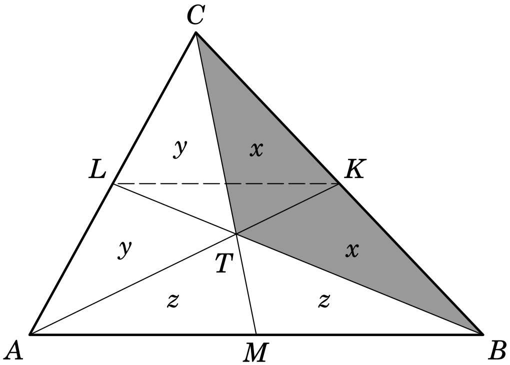
The right side of the equation is divisible by 25 and we therefore have $x = 25n$ for some $n \in \mathbb{N}$. Then $12n = y^2$ and the left side of the above equation is divisible by 6, i.e. $y = 6m$ for some $m \in \mathbb{N}$. We obtain $n = 3m^2$ and the right side is divisible by 3. Then $n = 3p$ for $p \in \mathbb{N}$ and $p = m^2$. Because the numbers x and y must be the smallest, we have $p = m = 1$.

Since $x = 25 \cdot n = 25 \cdot 3 \cdot p = 75$, $y = 6 \cdot m = 6$, the solution is

$$x = 75 \qquad \text{and} \qquad y = 6.$$

C–I–4

Let T be the centroid of the triangle ABC (see the picture) and K, L, M be the midpoints of its sides BC, CA, AB, respectively. Since the line segments BK and CK are congruent, the triangles BKT and CKT have the same area x. Similarly, the triangles CLT and ALT have the same area y, and the triangles AMT and BMT have the area z. Since $ABKL$ is a trapezoid we have $x = y$. Analogously we can prove $y = z$, which yields $x = y = z$ and the proof is finished.



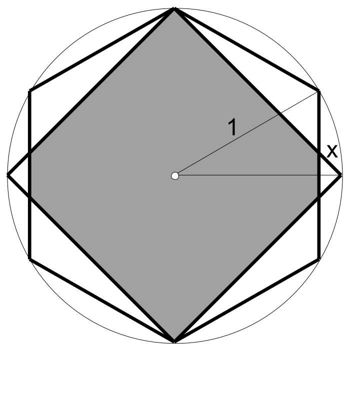
Category C (Team Competition)

C–T–1

The section in question results from the square by cutting off two right-angled isosceles triangles with side-length x. Its area is therefore equal to $2-x^2$.

In oder to determine x, we note that the altitude of the right-angled isosceles triangle is $1-\frac{\sqrt{3}}{2}$. We therefore have $x=\sqrt{2}\cdot\left(1-\frac{\sqrt{3}}{2}\right)$, and the area is equal to

$$2-\left(\sqrt{2}\cdot\left(1-\frac{\sqrt{3}}{2}\right)\right)^2=2\sqrt{3}-\frac{3}{2}.$$



C–T–2

Rewriting the right-hand side of the expresion V we obtain

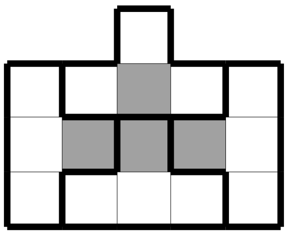
$$V = 4(a^2 + b^2 + c^2) - [(a + b)^2 + (b + c)^2 + (c + a)^2] =$$
$$= 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca =$$
$$= (a - b)^2 + (b - c)^2 + (c - a)^2.$$

It is clear that the last sum is non-negative (for any real numbers a, b, c). Moreover, $V = 0$ if and only if $a = b = c$.

C–T–3

The smallest number of cells is 4. If we colour the four cells as shown in the diagramm, any placement of the polyomino will cover one of these green cells.

No smaller number is possible, since the board can be cut into the four parts shown in the shape of the given polyomino, no two of which have a common cell.



Results

Category A (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Miroslav Klimoš	GMK Bílovec	8	8	8	8	32
2.	Jitka Novotná	GMK Bílovec	8	8	3	8	27
3.–4.	Daniel Steuber	BRG Kepler Graz	8	8	2	0	18
	Florian Andritsch	BRG Kepler Graz	8	8	1	1	18
5.–6.	Dagmar Plháková	GJŠ Přerov	8	5	3	0	16
	Lucie Mohelníková	GMK Bílovec	8	8	0	0	16
7.	Dita Přikrylová	GJŠ Přerov	8	1	6	0	15
8.–9.	Marcin Brożek	I LO Chorzów	8	0	3	0	11
	Katharina Albert	BRG Kepler Graz	8	0	3	0	11
10.–12.	Eliška Nekvapilová	GMK Bílovec	8	0	2	0	10
	Martin Ministr	G Olomouc–Hejčín	8	0	2	0	10
	Lukas Andritsch	BRG Kepler Graz	8	1	1	0	10
13.	Petra Chytilová	SG Olomouc	8	0	1	0	9
14.–16.	Alena Stránská	SG Olomouc	8	0	0	0	8
	Radosław Dudkiewicz	I LO Chorzów	8	0	0	0	8
	Jakub Dubovic	SG Olomouc	8	0	0	0	8
17.–19.	Josef Sedláček	GJŠ Přerov	1	0	1	0	2
	David Kraus	G Olomouc–Hejčín	2	0	0	0	2
	Lukáš Blísa	GJŠ Přerov	1	0	1	0	2
20.	Jakub Vrtný	G Olomouc–Hejčín	1	0	0	0	1
21.–22.	Jaroslav Pernica	SG Olomouc	0	0	0	0	0
	Jan Grygárek	G Olomouc–Hejčín	0	0	0	0	0

Category B (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.–3.	Martin Broušek	GJŠ Přerov	8	8	8	8	32
	Simona Domesová	GMK Bílovec	8	8	8	8	32
	Jakub Solovský	GMK Bílovec	8	8	8	8	32
4.	Joanna Kołodziej	I LO Chorzów	8	8	8	7	31
5.	Igor Sikora	I LO Chorzów	8	6	8	8	30
6.–7.	Paweł Gomoluch	I LO Chorzów	8	5	8	7	28
	Krzysztof Chrobak	I LO Chorzów	8	4	8	8	28
8.	Paweł Golda	I LO Chorzów	8	4	8	6	26
9.	Marie Kročová	GJŠ Přerov	1	3	8	8	20
10.–11.	Petr Boroš	GMK Bílovec	1	8	4	3	16
	Radim Dudek	G Olomouc–Hejčín	0	2	6	8	16
12.–13.	Adéla Indráková	G Olomouc–Hejčín	0	2	6	7	15
	Wojciech Lis	I LO Chorzów	8	0	2	5	15
14.	Simeon Kanya	BRG Kepler Graz	8	6	0	0	14
15.	Tamara Skokánková	G Olomouc–Hejčín	0	0	4	6	10
16.	Věra Kumová	GJŠ Přerov	0	8	0	1	9
17.	Jakub Ehrenberger	G Olomouc–Hejčín	3	2	0	2	7
18.	Josef Malík	GMK Bílovec	1	1	0	4	6
19.–20.	Martin Gutjahr	BRG Kepler Graz	0	0	2	2	4
	Christopher Schinnerl	BRG Kepler Graz	1	2	1	0	4
21.–22.	Zuzana Foltisová	SG Olomouc	0	1	0	2	3
	Anna Kubíčková	SG Olomouc	0	1	0	2	3
23.–24.	Barbora Benešová	SG Olomouc	0	2	0	0	2
	Christoph Schober	BRG Kepler Graz	0	0	0	2	2
25.	Tung Tran Thanh	GJŠ Přerov	0	1	0	0	1
26.	Renáta Heinzová	SG Olomouc	0	0	0	0	0

Category C (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.–2.	Tomasz Cieśla	G Nr 10 Chorzów	8	8	8	8	32
	Tomasz Depta	G Nr 10 Chorzów	8	8	8	8	32
3.	Marek Teuchner	GMK Bílovec	8	7	8	7	30
4.–5.	Bernd Prach	BRG Kepler Graz	7	7	8	6	28
	David Juřík	G Olomouc–Hejčín	4	8	8	8	28
6.	Patrycja Mrowiec	G Nr 11 Chorzów	8	4	8	6	26
7.	Rafał Kuzior	G Nr 10 Chorzów	8	8	8	0	24
8.	Klára Sládečková	GJŠ Přerov	3	4	8	8	23
9.	Stanislav Horák	GMK Bílovec	6	8	7	1	22
10.	Gabriela Olivíková	GJŠ Přerov	4	5	8	4	21
11.	Martin Kavík	GMK Bílovec	4	8	6	2	20
12.	Manuel Gruber	BRG Kepler Graz	6	4	8	0	18
13.–14.	Eva Gocníková	GJŠ Přerov	2	6	7	2	17
	Michaela Šmoldasová	SG Olomouc	5	8	4	0	17
15.	Clemens Andritsch	BRG Kepler Graz	0	8	8	0	16
16.	Zdeněk Přivřel	G Olomouc–Hejčín	1	6	8	0	15
17.	Florian Krach	BRG Kepler Graz	0	5	8	1	14
18.–19.	Kateřina Šimková	SG Olomouc	6	0	4	0	10
	Tomáš Majda	G Olomouc–Hejčín	3	6	1	0	10
20.	Simona Kapolková	GMK Bílovec	3	2	2	2	9
21.	Jakub Šuba	SG Olomouc	0	4	4	0	8
22.–23.	Kateřina Sluková	G Olomouc–Hejčín	2	2	0	2	6
	Roman Vrána	SG Olomouc	0	4	0	2	6
24.	David Halata	GJŠ Přerov	0	0	0	0	0

Category A (Team Competition)

Rank	School	1	2	3	∑
1.	Gymnázium Mikuláše Koperníka Bílovec	8	8	8	24
2.	BRG Kepler Graz	5	4	8	17
3.	Gymnázium Jakuba Škody Přerov	6	0	0	6
4.	I LO Chorzów	4	0	1	5
5.	Gymnázium Olomouc–Hejčín	2	1	0	3
6.	Slovanské gymnázium Olomouc	2	0	0	2

Category B (Team Competition)

Rank	School	1	2	3	∑
1.	Gymnázium Mikuláše Koperníka Bílovec	8	8	4	20
2.	Gymnázium Jakuba Škody Přerov	8	6	4	18
3.–4.	BRG Kepler Graz	7	6	0	13
	Gymnázium Olomouc–Hejčín	2	6	5	13
5.	I LO Chorzów	0	8	4	12
6.	Slovanské gymnázium Olomouc	0	5	0	5

Category C (Team Competition)

Rank	School	1	2	3	∑
1.	Gimnazjum Nr 10 Chorzów	8	8	8	24
2.	Gymnázium Mikuláše Koperníka Bílovec	6	4	8	18
3.	Gymnázium Jakuba Škody Přerov	3	4	8	15
4.	Slovanské gymnázium Olomouc	1	4	8	13
5.	BRG Kepler Graz	2	2	8	12
6.	Gymnázium Olomouc–Hejčín	0	0	8	8

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