Palacký University Olomouc, Faculty of Science

MATHEMATICAL DUEL '12

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Preface

The 20th International Mathematical Duel was held from 7–10 March 2012 in Bílovec. In this year the competition was organized by Nicolaus Copernicus Gymnasium Bílovec in cooperation with Faculty of Science of Palacký University Olomouc.

Seven school-teams from Austria, Czech Republic, Poland, Italy, Romania and Bulgaria took part in this traditional mathematical competition, namely from Bundesrealgymnasium Kepler, Graz, Gymnázium M. Koperníka, Bílovec, I Liceum Ogólnokształcące im. J. Słowackiego, Chorzów, Gymnázium J. Škody, Přerov, Liceo Scientifico Statale A. Labriola, Roma-Ostia, Colegiul National I. L. Caragiale, Ploiesti and for the first time teams from Sofijska matematičeska gimnazia Paisij Hilendarki, Sofia as guests.

As usual the competition was provided in the three categories (A – contestants of the last two years, B – contestants of the 5th and 6th years, and C – contestants of the 3rd and 4th years of eight-year grammar school). Twelve contestants (more precisely 4 in any category) of any school took part in this competition, i.e. 81 contestants in total.

This booklet contains all problems with solutions and results of the 20th International Mathematical Duel from the year 2012.

Authors

Problems

Category A (Individual Competition)

A–I–1

Solve in the domain of integers the following system of equations

$$x + \frac{2}{y} = z,$$
$$y + \frac{4}{z} = x,$$
$$z - \frac{6}{x} = y.$$

Jacek Uryga

A–I–2

We are given a cyclic quadrilateral $ABCD$ with $|\angle BDC| = |\angle CAD|$ and $|AB| = |AD|$. Prove that there exists a circle, which is tangent to all four sides of the quadrilateral $ABCD$.

Robert Geretschläger

A–I–3

Determine all cubic polynomials $P(x)$ with real coefficients such that the equation $P(x) = 0$ has three real roots (not necessarily different) fulfilling the following conditions:

a) The number 1 is a root of the considered equation.
b) For each root t of the equation $P(x) = 0$ the condition $P(2t) = t$ holds.

Pavel Calábek

A–I–4

Determine the minimum value of the expression

$$V = \frac{\sin\alpha}{\sin\beta\ \sin\gamma} + \frac{\sin\beta}{\sin\gamma\sin\alpha} + \frac{\sin\gamma}{\sin\alpha\ \sin\beta},$$

where α, β, γ are interior angles of a triangle.

Jaroslav Švrček

Category A (Team Competition)

A–T–1

Solve the following equation in positive integers

$$abc = 2a + 3b + 5c.$$

Pavel Calábek

A–T–2

Let us consider an acute-angled triangle ABC in the plane. Let D, E, F be the feet of altitudes from vertices A, B, C, respectively. Further, let K, L, M denote points of intersection of the lines AD, BE, CF with the circumcircle of the triangle ABC (different from the vertices A, B, C), respectively. Prove that the inequality

$$\min\left\{\frac{|KD|}{|AD|}, \frac{|LE|}{|BE|}, \frac{|MF|}{|CF|}\right\} \leq \frac{1}{3}$$

holds for all acute-angled triangles ABC.

Jaroslav Švrček

A–T–3

Peter's kit contains 6 identical sticks of 6 different colours. Peter can construct the model of a regular tetrahedron from these six sticks. How many distinct models exist?

Pavel Calábek

Category B (Individual Competition)

B–I–1

Determine all pairs (p, x) fulfilling the equation

$$x^2 = p^3 + 1.$$

where p is a prime and x is an integer.

Jaroslav Švrček

B–I–2

A parallelogram $ABCD$ is given in plane. A line ℓ passing through B meets the side CD at the point E and the ray AD at the point F. Determine the ratio of the areas of the triangles ABF and BEC in terms of the ratio $|CE| : |ED|$.

Jacek Uryga

B–I–3

Let k, n be arbitrary real numbers with $1 \leq k \leq n$. Prove that the inequality

$$k(n - k + 1) \geq n$$

holds. When does equality hold?

Józef Kalinowski

B–I–4

Prove that 2012 cannot be written as the sum of two perfect cubes. Is it possible to write 2012 as the difference of two perfect cubes? If not, prove that it is impossible.

Robert Geretschläger

Category B (Team Competition)

B–T–1

Determine all real polynomials $P(x)$, such that

$$P(P(x)) = x^4 + ax^2 + 2a$$

holds for some real number a.

Robert Geretschläger

B–T–2

We are given an isosceles right-angled triangle ABC. Let K be the midpoint of the hypotenuse AB of the given triangle. Find the set of vertices L of all isosceles right-angled triangles KLM with hypotenuse KL, such that the point M belongs to the side AC.

Jaroslav Švrček

B–T–3

Determine all triples (a, b, c) of positive integers for which each of the three numbers a, b, c is a divisor of the sum $a + b + c$.

Robert Geretschläger

Category C (Individual Competition)

C–I–1

Determine all positive integers such that the sum

$$\frac{x}{2} + \frac{2}{x}$$

is an integer.

Jaroslav Švrček

C–I–2

We are given a trapezoid $ABCD$ with $AB \parallel CD$, such that there exists a point E on the side BC with $|CE| = |CD|$ and $|BE| = |AB|$. Prove that AED is a right-angled triangle.

Józef Kalinowski

C–I–3

Two positive integers are called *friends* if

- ▷ each is composed of the same number of digits,
- ▷ the digits in one are in increasing order and the digits in the other are in decreasing order, and
- ▷ the two numbers have no digits in common (like for example the numbers 147 and 952).

Solve the following problems

a) Determine the number of all two-digit numbers that have a friend.
b) Determine the largest number that has a friend.

Robert Geretschläger

C–I–4

Let ABC be a right-angled triangle with the hypotenuse AB, such that $|AC| : |BC| = 2 : 3$ holds. Let D be the foot of its altitude from C. Determine the ratio $|AD| : |BD|$.

Józef Kalinowski

Category C (Team Competition)

C–T–1

Determine the number of all seven-digit numbers which are divisible by 4, such that the sum of all their digits is 4.

Józef Kalinowski

C–T–2

We are given a right-angled triangle ABC with right angle at C. A point D lies on AB, such that $|BD| = |BC|$. A point E lies on the line perpendicular to AB and passing through A, such that $|AE| = |AC|$. The points E and C are in the same half-plane defined by AB. Show that the points C, D and E lie on a common line.

Erich Windischbacher

C–T–3

We are given 8 coins, no two of which have the same weight, and a scale with which we can determine which group of coins placed on either end is heavier and which is lighter. We wish to determine which of the 8 coins is the heaviest and which is the lightest. Prove that this can be done with at most 10 weighings.

Robert Geretschläger

Solutions

Category A (Individual Competition)

A–I–1

Note that the fraction $\frac{2}{y}$ is an integer and therefore $y \in \{-2, -1, 1, 2\}$.

Since $x, y, z \neq 0$, we can equivalently multiply the equations by z, y and x, respectively, obtaining the system

$$\begin{aligned} xy + 2 &= yz, \\ yz + 4 &= zx, \\ zx - 6 &= xy. \end{aligned}$$

It is easy to see that the last equation results from the first two and therefore it can be omitted.

Now, we observe that if a triple of integers (x, y, z) is a solution of the system, then also the triple $(-x, -y, -z)$ is a solution. Thus, we have to consider only two cases for $y > 0$.

If $y = 1$, then we get

$$\begin{aligned} x + 2 &= z, \\ z + 4 &= zx. \end{aligned}$$

and so

$$\begin{aligned} x &= z - 2, \\ z + 4 &= z(z - 2). \end{aligned}$$

In this case we obtain two solutions: $z = -1$, $x = -3$ and $z = 4$, $x = 2$.

If $y = 2$, then we get

$$\begin{aligned} 2x + 2 &= 2z, \\ 2z + 4 &= zx. \end{aligned}$$

and then

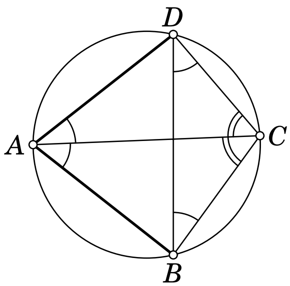
$$\begin{aligned} x &= z - 1, \\ 2z + 4 &= z(z - 1). \end{aligned}$$

In this case we obtain two another solutions: $z = -1$, $x = -2$ and $z = 4$, $x = 3$.

Conclusion. The complete solution consists of eight triples (x, y, z): $(-3, 1, -1)$, $(2, 1, 4)$, $(-2, 2, -1)$, $(3, 2, 4)$, $(3, -1, 1)$, $(-2, -1, -4)$, $(2, -2, 1)$, $(-3, -2, -4)$.

A–I–2

Since $|\angle CAD| = |\angle BDC| = |\angle BAC|$, we have $|BC| = |CD|$. It therefore follows that triangles ABC and ADC are congruent, since we have $|AB| = |AD|$, $|BC| = |BD|$ and AC is a common side. We see that



$ABCD$ is a kite (deltoid), and it therefore certainly has an incircle with midpoint on AC, as claimed.

A–I–3

If 1 is triple root of the $P(x) = 0$ then $P(x) = a(x-1)^3$ and from the condition b) we obtain $a \cdot 1^3 = 1$ and so $a = 1$ and in this case we have

$$P(x) = (x-1)^3. \qquad (1)$$

If 1 is double root of the $P(x) = 0$ then $P(x) = a(x-1)^2(x-x_1)$ where $x_1 \neq 1$. Using b) for $t = 1$ and $t = x_1$ we get

$$a(2-x_1) = 1 \qquad \text{and} \qquad a(2x_1-1)^2 x_1 = x_1.$$

From the first equation $a \neq 0$ and $x_1 = 2 - \frac{1}{a}$. The second equation gives us either $x_1 = 0$ (so $a = \frac{1}{2}$) or $(2x_1-1)^2 = \frac{1}{a}$. Substituting for $x_1 = 2 - \frac{1}{a}$ we get

$$9 - \frac{13}{a} + \frac{4}{a^2} = 0.$$

Solving this equation we obtain $a = 1$ and $a = \frac{4}{9}$. In these cases $x_1 = 1$ (which contradicts $x_1 \neq 1$) or $x_1 = -\frac{1}{4}$. So we get further solutions

$$P(x) = \frac{1}{2}x(x-1)^2 \qquad \text{and} \qquad P(x) = \frac{4}{9}(x-1)^2(x+\frac{1}{4}). \qquad (2)$$

If 1 is single root of the the $P(x) = 0$ and this equation has only one root $x_1 \neq 1$ then $P(x) = a(x-1)(x-x_1)^2$. Using b) we obtain

$$a(2-x_1)^2 = 1 \qquad \text{and} \qquad a(2x_1-1)x_1^2 = x_1.$$

From the first equation $a \neq 0$ and $\frac{1}{a} = (2-x_1)^2$. The second equation implies either $x_1 = 0$ (so $a = \frac{1}{4}$) or $2x_1^2 - x_1 = \frac{1}{a}$. Substituting for $\frac{1}{a} = (2-x_1)^2$ we get

$$x_1^2 + 3x_1 - 4 = 0.$$

Solving this equation we obtain $x_1 = 1$ (which contradicts $x_1 \neq 1$) and $x_1 = -4$, so $a = \frac{1}{36}$. In this case we get further solutions

$$P(x) = \frac{1}{4}x^2(x-1) \qquad \text{and} \qquad P(x) = \frac{1}{36}(x-1)(x+4)^2. \tag{3}$$

Finally, let us assume that the cubic equation has three distinct roots and $P(x) = ax^3 + bx^2 + cx + d$. From b) follows that an equation $P(2x) - x = 0$ has the same three distinct roots. This implies that the equation $0 = 8P(x) - (P(2x) - x) = 4bx^2 + (6c+1)x + 7d$ has the same three distinct roots and so its coefficients vanish. Easily we get $b = 0$, $c = -\frac{1}{6}$, $d = 0$. The condition a) implies that 1 is root and so $a - \frac{1}{6} = 0$. From it follows that $a = \frac{1}{6}$ and in this case we get the solution

$$P(x) = \frac{1}{6}(x^3 - x) = \frac{1}{6}(x-1)\,x\,(x+1). \tag{4}$$

Conclusion. The given equation has exactly six solutions given by (1)–(4).

A–I–4

Since α, β and γ are interior angles of some triangle, the values $\sin\alpha$, $\sin\beta$ and $\sin\gamma$ are positive real numbers. We can use the A-G means inequality in the form

$$V = \frac{\sin\alpha}{\sin\beta\,\sin\gamma} + \frac{\sin\beta}{\sin\gamma\sin\alpha} + \frac{\sin\gamma}{\sin\alpha\,\sin\beta} \geq 3\sqrt[3]{\frac{1}{\sin\alpha\,\sin\beta\,\sin\gamma}}.$$

Since the function $\sin x$ is concave on $(0;\pi)$, we can estimate the denominator of the right-hand side of the last inequality by Jensen's inequality (combining with the A-G means inequality) in the following way:

$$\begin{aligned}\sqrt[3]{\sin\alpha\ \sin\beta\ \sin\gamma} &\le \frac{\sin\alpha+\sin\beta+\sin\gamma}{3}\\ &\le \sin\left(\frac{\alpha+\beta+\gamma}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.\end{aligned}$$

Thus we have $V \ge 2\sqrt{3}$ with equality for $\alpha = \beta = \gamma = \frac{1}{3}\pi$, i.e. in the case of the equilateral triangle.

Conclusion. The minimum value of the expression V is $2\sqrt{3}$.

Another solution (by Václav Kapsia, GMK Bílovec). Let a, b, c be lengths of sides and P the area of the triangle. Rewriting the expression V and using the law of sines we obtain

$$\begin{aligned}V &= \frac{\sin^2\alpha}{\sin\alpha\ \sin\beta\ \sin\gamma} + \frac{\sin^2\beta}{\sin\alpha\ \sin\beta\ \sin\gamma} + \frac{\sin^2\gamma}{\sin\alpha\ \sin\beta\ \sin\gamma}\\ &= \frac{a^2}{bc\sin\alpha} + \frac{b^2}{ca\sin\beta} + \frac{c^2}{ab\sin\gamma}.\end{aligned}$$

Since

$$bc\sin\alpha = ca\sin\beta = ab\sin\gamma = 2P,$$

we have

$$V = \frac{a^2+b^2+c^2}{2P}.$$

By the well-known inequality $a^2+b^2+c^2 \ge 4\sqrt{3}\,P$ (see for example the problem 2, 3rd IMO), we finally get the required inequality $V \ge 2\sqrt{3}$.

Another solution. Using the A-G means inequality we have

$$\frac{\sin\alpha}{\sin\beta\ \sin\gamma} + \frac{\sin\beta}{\sin\gamma\ \sin\alpha} \ge \frac{2}{\sin\gamma}, \qquad \frac{\sin\beta}{\sin\gamma\ \sin\alpha} + \frac{\sin\gamma}{\sin\alpha\ \sin\beta} \ge \frac{2}{\sin\alpha},$$

$$\frac{\sin\gamma}{\sin\alpha\ \sin\beta} + \frac{\sin\alpha}{\sin\beta\ \sin\gamma} \ge \frac{2}{\sin\beta}.$$

Adding up all three inequalities and using Jensen's inequality for the convex function $\frac{1}{\sin x}$ on $(0;\pi)$ we obtain

$$V \geq \frac{1}{\sin\alpha} + \frac{1}{\sin\beta} + \frac{1}{\sin\gamma} \geq \frac{3}{\sin\left(\frac{\alpha+\beta+\gamma}{3}\right)} = 2\sqrt{3},$$

which is required.

Category A (Team Competition)

A–T–1

Firstly we discuss some special cases for a and b.

For $a = 1$ we have to solve $bc = 2 + 3b + 5c$ which is equivalent to $(b-5)(c-3) = 17$. Since $b - 5 > -4$ and $c - 3 > -2$, both factors are positive integers factors of the prime 17, so we have solutions

$$(a,b,c) \in \{(1,6,20);(1,22,4)\}.$$

For $a = 2$ we have the equation $2bc = 4 + 3b + 5c$ which is equivalent to $(2b-5)(2c-3) = 23$. In the same way we obtain solutions

$$(a,b,c) \in \{(2,3,13);(2,14,2)\}.$$

For $b = 1$ we have to solve $ac = 2a + 3 + 5c$ which is equivalent to $(a-5)(c-2) = 13$. Further solutions are then

$$(a,b,c) \in \{(6,1,15);(18,1,3)\}.$$

For $b = 2$ we have the equation $2ac = 2a + 6 + 5c$ which is equivalent to $(2a-5)(c-1) = 11$. Thus we have solutions

$$(a,b,c) \in \{(3,2,12);(8,2,2)\}.$$

Now we can assume $a, b \geq 3$. In this case $a - \frac{5}{b} \geq \frac{4}{3}$ and $b - \frac{5}{a} \geq \frac{4}{3}$. From $abc = 2a + 3b + 5c$ we obtain

$$c = \frac{2a}{ab-5} + \frac{3b}{ab-5} = \frac{2}{b-\frac{5}{a}} + \frac{3}{a-\frac{5}{b}} \leq 2\cdot\frac{3}{4} + 3\cdot\frac{3}{4} = \frac{15}{4}.$$

So $c \le 3$.

Finally we have to discuss three possibilities for c.

- ▹ If $c = 1$ we will solve the equation $ab = 2a + 3b + 5$ which is equivalent to $(a-3)(b-2) = 11$. Now we obtain two solutions

$$(a, b, c) \in \{(4, 13, 1); (14, 3, 1)\}.$$

- ▹ If $c = 2$ we will solve the equation $2ab = 2a + 3b + 10$ which is equivalent to $(2a-3)(b-1) = 13$. In this case we get

$$(a, b, c) \in \{(2, 14, 2); (8, 2, 2)\},$$

but both of them have been obtained previously.

- ▹ If $c = 3$ we will solve the equation $3ab = 2a + 3b + 15$ which is equivalent to $(a-1)(3b-2) = 17$. In this case we have only one solution in positive integers

$$(a, b, c) \in \{(18, 1, 3)\}$$

also obtained previously.

Conclusion. All solutions of the original equation in positive integers are triples (a, b, c) from the set

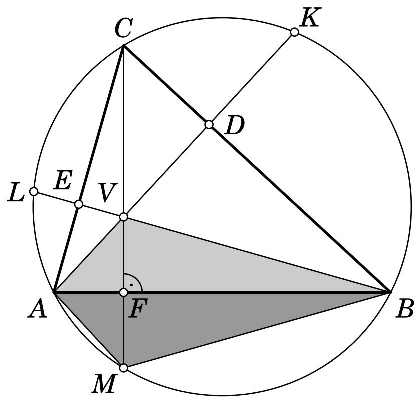
$$\{(1, 6, 20); (1, 22, 4); (2, 3, 13); (2, 14, 2); (3, 2, 12);$$
$$(4, 13, 1); (6, 1, 15); (8, 2, 2); (14, 3, 1); (18, 1, 3)\}.$$

A–T–2

Let V be the orthocenter of the considered acute-angled triangle ABC. It is well-known that the mirror images of V by the lines AB, BC a CA lie on the cirmumcircle of this triangle. There are the points M, K a L, respectively. Thus for instance, the triangle ABV is congruent to the triangle ABM (both triangles have the same area).

For the areas S_{ABV} and S_{ABC} of the triangles ABV and ABC, respectively, we have

$$\frac{S_{ABV}}{S_{ABC}} = \frac{|VF|}{|CF|} = \frac{|MF|}{|CF|}.$$



Similarly

$$\frac{S_{BCV}}{S_{ABC}} = \frac{|VD|}{|AD|} = \frac{|KD|}{|AD|} \qquad \text{and} \qquad \frac{S_{CAV}}{S_{ABC}} = \frac{|VE|}{|BE|} = \frac{|LE|}{|BE|}.$$

Adding up left sides the last three equalities we get

$$\frac{S_{ABV}}{S_{ABC}} + \frac{S_{BCV}}{S_{ABC}} + \frac{S_{CAV}}{S_{ABC}} = \frac{S_{ABV} + S_{BCV} + S_{CAV}}{S_{ABC}} = 1.$$

From the other side we also have

$$\frac{|KD|}{|AD|} + \frac{|LE|}{|BE|} + \frac{|MF|}{|CF|} = 1.$$

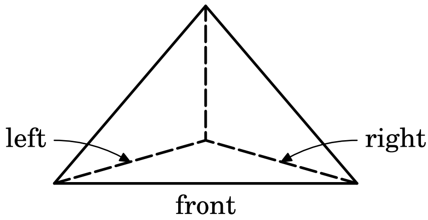
Since each of three fractions (summands) on the left side of the last expression is a positive real number, the validity of the given statement immediately follows, i.e. the inequality

$$\min\left\{\frac{|KD|}{|AD|}, \frac{|LE|}{|BE|}, \frac{|MF|}{|CF|}\right\} \leq \frac{1}{3}$$

is true and the proof is finished.

A–T–3

Let us call the coloured sticks (edges) by $1, 2, \ldots, 6$. Further let us put the model of the tetrahedron such that the stick 1 is on the desk in front of us. Other edges on the desk let us call left and right.



Let us consider the stick 2. If this edge is skew to stick 1 we can rotate the tetrahedron about the common axis of edges 1 and 2 such that the stick 3 and the stick 1 are on the desk. The edge 3 is either the left or right edge and we cannot rotate the tetrahedron in such a way that the left edge 3 moves to the right edge. Now the tetrahedron is fixed. There are 3! ways to complete colouring the other edges and so in this case we have $2 \cdot 3! = 12$ distinct colourings depending on whether the edge 3 is left or right.

Now we suppose, that the stick 2 is a neighbour of the stick 1. We rotate the tetrahedron such that the edges 1 and 2 are on the desk and stick 1 is in front. Now the tetrahedron is fixed and we have 4! ways to complete colourings other edges. In this case we have $2 \cdot 4! = 48$ distinct colourings. Altogether, there are $12 + 48 = 60$ distinct colourings of the edges of the regular tetrahedron by 6 colours.

Category B (Individual Competition)

B–I–1

First of all, we can rewrite and factorize the given equation to the form

$$x^2 - 1 = (x - 1)(x + 1) = p^3.$$

With each solution (x, p) of the given equation $(-x, p)$ is also a solution. Therefore we can consider $x \geq 0$ and (with $x-1 < x+1$) we have two possibilities in that case:

- ▷ $(x-1=1) \wedge (x+1=p^3)$. This implies $x=2$, $p^3=3$ and we therefore have no solution in this case.
- ▷ $(x-1=p) \wedge (x+1=p^2)$. Subtracting these two equations we obtain the following quadratic equation with unknown p

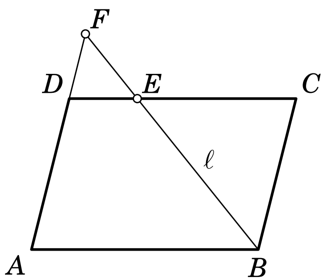
$$p^2 - p - 2 = 0$$

with two real roots $p=-1$, which doesn't fulfill the conditions of the given problem, and further $p=2$.

Conclusion. The given equation has exactly two solutions, namely: $(x, p) = (3; 2)$ and $(x, p) = (-3; 2)$.

B–I–2

Note that by the angle-angle rule the triangles ABF and BEC are similar (the angles $|\angle ABF| = |\angle BEC|$ and $|\angle AFB| = |\angle CBE|$ are alternate ones). Therefore the ratio of the areas of the triangles ABF and



BEC is equal to the square of the ratios of corresponding sides. Thus

$$\frac{S_{ABF}}{S_{BEC}} = s^2, \qquad \text{where} \qquad s = \frac{|AB|}{|CE|}.$$

Since $ABCD$ is parallelogram, we get

$$s = \frac{|CD|}{|CE|} = \frac{|CE| + |ED|}{|CE|} = 1 + \frac{|ED|}{|CE|}$$

Denoting $r = \frac{|CE|}{|ED|}$, we have

$$\frac{S_{ABF}}{S_{BEC}} = \left(1 + \frac{1}{r}\right)^2 = \left(\frac{r+1}{r}\right)^2.$$

B–I–3

Rewriting the given inequality $k(n-k+1) \geq n$ in the equivalent form, we obtain $(n-k)(k-1) \geq 0$. The last inequality is true by the assumptions $1 \leq k \leq n$.

Equality holds for $k = n$ or $k = 1$.

B–I–4

Firstly we will prove that every cube of an integer has remainder 0, 1 or 8 after division by 9. Let $n = 3k + r$, where $r \in \{0, 1, 2\}$ and k is an integer. This follows from the identity $n^3 = 9(3k^3 + 3k^2r + kr^2) + r^3$.

This implies that the sum of two perfect cubes has remainders 0, 1, 2, 7 or 8. Since 2012 has the remainder 5 after division by 9, it follows that 2012 cannot be expressed as the sum of two perfect cubes.

This also means that 2012 cannot be expressed as the difference of two perfect cubes because $m^3 - n^3 = m^3 + (-n)^3$.

Category B (Team Competition)

B–T–1

Since $P(P(x))$ is of fourth degree, $P(x)$ must be quadratic, and we have $P(x) = x^2 + px + q$. From this we obtain

$$\begin{aligned} P(P(x)) &= (x^2 + px + q)^2 + p(x^2 + px + q) + q \\ &= x^4 + 2px^3 + (p^2 + p + 2q)x^2 + (p^2 + 2pq)x + pq + q + q^2. \end{aligned}$$

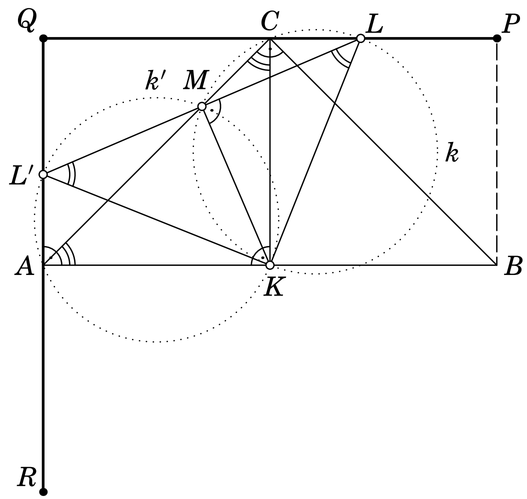
If this is to be equal to $x^4 + ax^2 + 2a$ for all values of x, we see that $p = 0$ must hold (by checking the cubic coefficient), and we therefore obtain $P(P(x)) = x^4 + 2qx^2 + q + q^2$. Comparing coefficients yields the quadratic equation $q^2 + q = 4q$, which is equivalent to $q^2 = 3q$, and q is therefore either equal to 0 or 3. The two possible polynomials are therefore $P(x) = x^2$ and $P(x) = x^2 + 3$.

B–T–2

Solution. Let the points M, L be situated as on the picture below. Since $|\angle KLM| = |\angle KCM| = 45^\circ$, the points K, L, C, M lie on the same circle k. Thus

$$|\angle KCL| = |\angle KML| = 90^\circ,$$

which means that k is a Thales circle with diameter KL and therefore the point L lies on the perpendicular to CK. If both points M, L are lying in the opposite half-plane defined by KC, we get the same result.



Similarly, let us consider the vertex $L = L'$ in the opposite half-plane defined by the line KM (see picture). Then the points K, M, L', A lie also on the Thales circle k' with the diameter KL'. Moreover both circles k and k' are congruent (their diameters KL and KL' are congruent).

This implies that the point L lies necessarily on the segment PQ *or* on the segment QR.

Conversely, it is easy to see that for an arbitrary point L which belongs to the broken line PQR there (uniquely) exists a point M on the side AC such that the triangle KLM is an right-angled isosceles triangle with hypotenuse KL.

Conclusion. The set of all points L with the required property is the broken line PRQ, such that $PQ \perp QR$ and C, A are the midpoints of the line segments PQ, QR, respectively.

B–T–3

Without loss of generality, we assume $a \le b \le c$. Since $c \mid (a+b+c)$, we have $c \mid (a+b)$ and therefore $c \le a+b \le 2c$ so $c = a+b$ or $2c = a+b$.

If $c = a+b$ from $b \mid (a+c) = 2a+b$ it follows $b \mid 2a$. This implies either $b = a$ or $b = 2a$. The first case gives triple $(a, a, 2a)$, the second one gives triple $(a, 2a, 3a)$.

If $2c = a+b$ from the inequality $a \le b \le c$ further it follows $a = b = c$.

Conclusion. There are 10 possible triples satisfying the problem. There are (a,a,a), $(a,a,2a)$, $(a,2a,a)$, $(2a,a,a)$, $(a,2a,3a)$, $(a,3a,2a)$, $(2a,a,3a)$, $(2a,3a,a)$, $(3a,a,2a)$, $(3a,2a,a)$ where a is an arbitrary positive integer what we can easy check.

Category C (Individual Competition)

C–I–1

Since

$$\frac{x}{2} + \frac{2}{x} = \frac{x^2 + 4}{2x},$$

we can see that x must be an even positive number (the denominator of the fraction on the right side is divisible by 2). Therefore $x = 2m$ (m is a positive integer). Further we can rewrite the given sum in the following form

$$\frac{x}{2} + \frac{2}{x} = \frac{4m^2 + 4}{4m} = \frac{m^2 + 1}{m} = m + \frac{1}{m},$$

which implies $m = 1$, and subsequently $x = 2$.

Conclusion. There exists only one positive integer x fulfilling conditions of the given problem, namely $x = 2$.

Another solution (by Jan Gocník, GJŠ Přerov). Let

$$\frac{2}{x} + \frac{x}{2} = n,$$

where n is an integer. Multiplying both sides of this equation by 2, we get

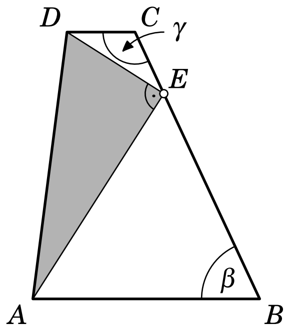
$$x + \frac{4}{x} = 2n.$$

Since x and $2n$ are positive integers, the number 4 must be divisible by x. Thus $x \in \{1, 2, 4\}$ and simultaneously $x + \frac{4}{x}$ must be even. Therefore $x = 2$.

C–I–2

Let $|\angle ABC| = \beta$ and $|\angle BCD| = \gamma$. In the trapezoid $ABCD$ we have $\beta + \gamma = 180^\circ$. Since $|CE| = |CD|$, the triangle DEC is isosceles and

$$|\angle CDE| = |\angle CED| = 90^\circ - \frac{\gamma}{2}.$$



Similarly, the triangle ABE is also isosceles and we have

$$|\angle EAB| = |\angle BEA| = 90^\circ - \frac{\beta}{2}.$$

and therefore the equality

$$|\angle CED| + |\angle DEA| + |\angle BEA| = 180^\circ$$

holds. Finally

$$90^\circ - \frac{\gamma}{2} + |\angle DEA| + 90^\circ - \frac{\beta}{2} = 180^\circ$$

and we obtain

$$|\angle DEA| = \frac{\beta}{2} + \frac{\gamma}{2} = \frac{\beta + \gamma}{2} = \frac{180^\circ}{2} = 90^\circ.$$

Thus AED is a right-angled triangle, and the proof is finished.

C–I–3

a) Every two-digit number n which is composed from different digits, has its digits in increasing or decreasing order. Moreover there are at least two non-zero digits a and b different from the digits of n. It follows, that the friend of n is one of numbers $\overline{ab}$ or $\overline{ba}$. So, the number of all two-digit numbers with a friend is equal to the number of all two-digit numbers composed of different digits. There are 90 two-digit number of which 9 $(11, 22, \ldots, 99)$ consist of identical digits. Therefore there are 81 two-digit numbers which have a friend.

b) If the number with a friend has k digits, its friend also has k different digits and together the have $2k$ different digits. Since there are 10 digits, the largest number with a friend has at most 5 digits. No number begins with 0, so 0 is in a number with digits in decreasing order. Moreover, if number n with digits in increasing order has a friend k, its palindrome is greater and has a friend (namely the palindrome to k). The largest number with a friend has different digits in decreasing order, has at most five digits and one of its digits is 0. So, the largest such number is therefore 98760 and its friend is 12345.

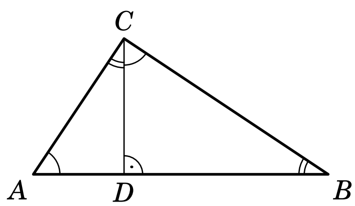
C–I–4

Let us consider a right-angled triangle ABC with hypotenuse AB such that $|AC| : |BC| = 2 : 3$. The right-angled triangles ADC and CDB are similar, because theirs measures of interior angles are equal. Then it holds

$$\frac{|AD|}{|CD|} = \frac{|CD|}{|DB|} = \frac{|AC|}{|BC|} = \frac{2}{3}.$$

This implies

$$|AD| = \frac{2}{3}\,|CD| \qquad \text{and} \qquad |DB| = \frac{3}{2}\,|CD|.$$



Further we obtain

$$\frac{|AD|}{|BD|} = \frac{\frac{2}{3}\,|CD|}{\frac{3}{2}\,|CD|} = \frac{4}{9},$$

and thus

$$|AD| : |BD| = 4 : 9.$$

Another solution. By Euclid's theorem in the right-angled triangle ABC we have

$$|AD| \cdot |AB| = |AC|^2, \qquad |BD| \cdot |AB| = |BC|^2,$$

which implies

$$|AD| : |BD| = |AC|^2 : |BC|^2 = 4 : 9.$$

Category C (Team Competition)

C–T–1

Let us consider four possibilities (by the first digit from the left) for seven-digit positive integers which are divisible by 4:

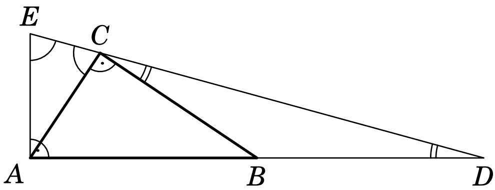
- ▷ The first digit from the left is 4. In this case only the number 4 000 000 fulfils both requirements of the given problem.
- ▷ The first digit from the left is 3. Then both assumptions are fulfilled by the numbers 3 100 000, 3 010 000, 3 001 000 and 3 000 100.
- ▷ The first digit from the left is 2. In this case both assumptions are fulfilled by positive integers 2 200 000, 2 020 000, 2 002 000, 2 000 200 and 2 000 020. Further we also obtain 2 110 000, 2 101 000, 2 100 100, 2 011 000, 2 010 100 and 2 001 100.
- ▷ The first digit from the left is 1. Then the assumptions are fulfilled by the numbers 1 300 000, 1 030 000, 1 003 000 and 1 000 300, also we obtain 1 210 000, 1 201 000, 1 200 100, 1 021 000, 1 020 100, 1 002 100 and 1 120 000, 1 102 000, 1 100 200, 1 100 020, 1 012 000, 1 010 200, 1 010 020, 1 001 200, 1 001 020, 1 000 120, 1 000 012 and finally we obtain the four numbers 1 111 000, 1 110 100, 1 101 100 and 1 011 100.

Conclusion. Altogether we have 41 positive integers fulfilling both requirements of the given problem.

C–T–2

Let $\alpha = |\angle BAC|$ and $\beta = |\angle ABC|$. Since both triangles BCD and ACE are isosceles, we can see that $|\angle BCD| = |\angle BDC| = \frac{\beta}{2}$, and $|\angle ACE| = |\angle AEC| = 45^\circ + \frac{\alpha}{2}$, . It therefore follows

$$|\angle ACE| + |\angle ACB| + |\angle BCD| = \left(45^\circ + \frac{\alpha}{2}\right) + 90^\circ + \frac{\beta}{2} = 135^\circ + \frac{\alpha + \beta}{2}.$$



Since $\alpha + \beta = 90^\circ$, the right-hand side of the last expression is equal to 180°. It follows that D, C and E are collinear, as claimed.

C–T–3

We first divide into 4 pairs and perform 4 weighings. The four heavier coins are put into group A, which must contain the heaviest coin, and the others in group B, which must contain the lightest. Dividing group A into 2 pairs, we repeat this, identifying the two heaviest coins, and one final weighing of the resulting pair identifies the heaviest coin in the group. We have made 3 weighings in group A to identify the heaviest coin, and three analogous weighings in group B identify the lightest. Altogether, we have performed 10 weighings, and identified the heaviest and lightest coin, as required.

Results

Category A (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Konstantinov Hristov Nikola	SMGPH Sofia	8	8	5	7	**28**
2.	Voroneanu Radu Ştefan	CNC Ploiesti	8	8	2	8	**26**
3.	Cangea Cătălina	CNC Ploiesti	8	7	2	8	**25**
4.	Kapsia Václav	GMK Bílovec	8	8	0	7	**23**
5.	Bodzilov Asenov Ivan	SMGPH Sofia	8	8	1	2	**19**
6.	Bungiu Alexandru Ionuţ	CNC Ploiesti	8	8	2	0	**18**
	Gocníková Eva	GJŠ Přerov	8	8	2	0	**18**
	Marinov Vanislavov Teodor	SMGPH Sofia	7	8	2	1	**18**
9.	Kortezov Ivajlov Ivo	SMGPH Sofia	6	8	0	1	**15**
10.	Kopf Michal	GMK Bílovec	3	8	1	2	**14**
11.	Andritsch Clemens	BRG Graz	3	8	1	1	**13**
12.	Solovská Kateřina	GMK Bílovec	4	8	0	0	**12**
13.	Trutman Pavel	GMK Bílovec	8	0	3	0	**11**
14.	Veigang-Rădulescu Vlad Petru	CNC Ploiesti	8	0	1	1	**10**
15.	Chmela Lukáš	GJŠ Přerov	0	8	1	0	**9**
16.	Prach Bernd	BRG Graz	3	0	5	0	**8**
	Weiss Andreas	BRG Graz	0	8	0	0	**8**
18.	Svibic Martina	BRG Graz	0	4	0	0	**4**
	Setlak Natalia	I LO Chorzów	4	0	0	0	**4**
20.	Harlenderová Alena	GJŠ Přerov	2	0	1	0	**3**
	Bastianelli Marianna	LSSL Roma	3	0	0	0	**3**
	Tobia Michele	LSSL Roma	0	3	0	0	**3**
23.	Simeoni Lorenzo	LSSL Roma	0	1	0	1	**2**
24.	Spyra Adam	I LO Chorzów	0	0	0	0	**0**
	Wernicki Wojciech	I LO Chorzów	0	0	0	0	**0**
	Krčmář Ondřej	GJŠ Přerov	0	0	0	0	**0**
	Centini Manuel	LSSL Roma	0	0	0	0	**0**

Category B (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Ivanova Todorova Velina	SMGPH Sofia	8	8	8	8	**32**
	Rogachev Ivanov Emilian	SMGPH Sofia	8	8	8	8	**32**
	Suvandzieva Rumenova Vladimira	SMGPH Sofia	8	8	8	8	**32**
4.	Paraschiv George	CNC Ploiesti	8	8	8	7	**31**
	Tenev Antonov Aleksandar	SMGPH Sofia	7	8	8	8	**31**
6.	Ławniczak Łukasz	I LO Chorzów	6	8	8	8	**30**
	Socha Jarosłav	I LO Chorzów	8	6	8	8	**30**
8.	Roşu Octavian	CNC Ploiesti	8	8	8	4	**28**
9.	Calábková Markéta	GJŠ Přerov	8	8	8	3	**27**
10.	Cremarenko Diana	CNC Ploiesti	2	8	8	8	**26**
11.	Kremel Tomáš	GJŠ Přerov	4	8	8	5	**25**
12.	Vincena Petr	GJŠ Přerov	8	7	8	0	**23**
13.	Prach Heinz	BRG Graz	8	0	8	4	**20**
14.	Knob Lukáš	GJŠ Přerov	8	8	3	0	**19**
15.	Wantula Szymon	GMK Bílovec	4	0	8	4	**16**
	Minorczyk Artur	I LO Chorzów	4	0	8	4	**16**
17.	Pudda Francesco	LSSL Roma	8	0	5	0	**13**
18.	Vaněk Petr	GMK Bílovec	3	2	7	0	**12**
19.	Krejčí Jan	GMK Bílovec	3	0	8	0	**11**
	Šrůtek Michal	GMK Bílovec	2	0	8	1	**11**
21.	Matei Andrei	CNC Ploiesti	0	8	1	0	**9**
22.	Prach Gerda	BRG Graz	4	0	2	2	**8**
23.	Mazziti Paolo	LSSL Roma	4	0	1	0	**5**
24.	Bordoni Simone	LSSL Roma	1	2	0	0	**3**
25.	Feistritzer Felix	BRG Graz	0	0	0	1	**1**
26.	Costantini Federico	LSSL Roma	1	0	0	0	**1**

Category C (Individual Competition)

Rank	**Name**	**School**	**1**	**2**	**3**	**4**	**∑**
1.	Andritsch Benedikt	BRG Graz	8	8	8	8	**32**
	Atanasov Raikov Daniel	SMGPH Sofia	8	8	8	8	**32**
	Markova Hristova Denica	SMGPH Sofia	8	8	8	8	**32**
	Najdenova Nikolaeva Violeta	SMGPH Sofia	8	8	8	8	**32**
5.	Gocník Jan	GJŠ Přerov	8	8	7	8	**31**
6.	Nicolescu Răzvan	CNC Ploiesti	8	8	5	8	**29**
	Rudzev Zdravkov Dimitar	SMGPH Sofia	5	8	8	8	**29**
8.	Borówka Sebastian	I LO Chorzów	4	8	8	8	**28**
	Tudor Costin	CNC Ploiesti	8	8	4	8	**28**
10.	Savu Mihnea	CNC Ploiesti	1	8	8	8	**25**
	Greco Giacomo	LSSL Roma	1	8	8	8	**25**
12.	Cappuccio Daniele	LSSL Roma	7	8	8	0	**23**
13.	Paliga Jakub	I LO Chorzów	7	8	4	3	**22**
14.	Ślusarczyk Michał	I LO Chorzów	8	0	5	8	**21**
15.	Andritsch Konstantin	BRG Graz	8	2	8	0	**18**
	Prach Doris	BRG Graz	8	0	2	8	**18**
17.	Vyciślok Artur	I LO Chorzów	1	8	8	0	**17**
18.	Horiatakis Daniel	BRG Graz	1	8	7	0	**16**
	Poljak Marian	GJŠ Přerov	1	1	8	6	**16**
20.	Mihalcea-Simoiu Theodor	CNC Ploiesti	1	2	4	8	**15**
21.	Marras Gloria	LSSL Roma	1	8	4	0	**13**
22.	Tížková Tereza	Bílovec	1	8	0	0	**9**
23.	Čáp Šimon	Bílovec	1	0	6	0	**7**
24.	Ferrante Giacomo	LSSL Roma	1	0	4	0	**5**
25.	Vojkůková Kateřina	Bílovec	1	0	1	0	**2**
	Vaculová Petra	GJŠ Přerov	1	1	0	0	**2**
27.	Novák Radek	Bílovec	1	0	0	0	**1**
	Andrlík Jiří	GJŠ Přerov	1	0	0	0	**1**

Category A (Team Competition)

Rank	School	1	2	3	∑
1.	CN Caragiale Ploiesti	8	8	8	**24**
2.	SMG PH Sofia	0	8	5	**13**
3.	BRG Graz	3	0	8	**11**
	GJŠ Přerov	3	0	8	**11**
	GMK Bílovec	1	6	4	**11**
6.	I LO Chorzów	3	0	0	**3**
7.	LSS Labriola Roma	0	0	2	**2**

Category B (Team Competition)

Rank	School	1	2	3	∑
1.	SMG PH Sofia	8	8	8	**24**
2.	I LO Chorzów	8	2	7	**17**
3.	LSS Labriola Roma	8	2	1	**11**
4.	BRG Graz	0	6	4	**10**
	CN Caragiale Ploiesti	1	1	8	**10**
6.	GMK Bílovec	0	2	4	**6**
7.	GJŠ Přerov	0	4	0	**4**

Category C (Team Competition)

Rank	School	1	2	3	∑
1.	CN Caragiale Ploiesti	8	8	8	**24**
	GJŠ Přerov	8	8	8	**24**
	I LO Chorzów	8	8	8	**24**
	SMG PH Sofia	8	8	8	**24**
5.	BRG Graz	8	3	8	**19**
6.	LSS Labriola Roma	7	3	8	**18**
7.	GMK Bílovec	8	0	8	**16**

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