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Jakub Škoda Gymnázium Přerov

MATHEMATICAL DUEL '14

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Preface

The 22nd Mathematical Duel was held from 12–15 March 2014 in Přerov. In this year the competition was organized by Gymnázium J. Škody in Přerov, mainly by Jan Raška, headmaster of the school.

Five school-teams from Austria, Czech Republic, Poland and Bulgaria took part in this traditional mathematical competition, namely from Bundesrealgymnasium Kepler, Graz, Gymnázium M. Koperníka, Bílovec, I Liceum Ogólnokształcące im. J. Słowackiego, Chorzów, Gymnázium J. Škody, Přerov and Sofijska matematičeska gimnazia Paisij Hilendarski, Sofia, as guests.

As usual the competition was provided in the three categories (A – contestants of the last two years, B – contestants of the 5th and 6th years, and C – contestants of the 3rd and 4th years of eight-year grammar school). Twelve contestants (more precisely 4 in any category) of any school took part in this competition, i.e. 59 contestants in total.

This booklet contains all problems with solutions and results of the 22nd Mathematical Duel from the year 2014.

Authors

Problems

Category A (Individual Competition)

A–I–1

Let $a \neq 0, b, c$ be real numbers with $|a + c| < |a - c|$. Prove that the quadratic equation $ax^2 + bx + c = 0$ has two real roots such that one of them is a positive and the second one is a negative number.

Jacek Uryga

A–I–2

We are given a tetrahedron $ABCD$ with pairwise perpendicular edges at the vertex D. Let K, L, M be midpoints of the egdes AB, BC, CA. Prove that the sum of the measures of angles at the vertex D in three adjacent faces of the tetrahedron $KLMD$ is equal to 180°.

Jaroslav Švrček

A–I–3

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real function satisfying

$$f(f(x) - y) = x - f(y) \qquad \text{for all } x, y \in \mathbb{R}.$$

Prove that f is an odd function, i.e. $f(-x) = -f(x)$ holds for all $x \in \mathbb{R}$.

Jacek Uryga

A–I–4

Determine all positive integers n for which there exist mutually distinct positive integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1.$$

Pavel Calábek

Category A (Team Competition)

A–T–1

Prove that the inequality

$$(a+9)\left(a^2+\frac{1}{a}+\frac{b^2}{8}\right) \geq (a+b+1)^2$$

holds for all positive real values of a and b. When does equality hold?

Robert Geretschläger

A–T–2

Red, green and blue fireflies live on the magic meadow. If

a) 2 blue fireflies meet, they change to 1 red firefly,
b) 3 red fireflies meet, they change to 1 blue firefly,
c) 3 green fireflies meet, they change to 1 red firefly,
d) 1 red and 1 green firefly meet, they change to 1 blue firefly,
e) 1 red and 1 blue firefly meet, they change to 2 green fireflies.

Initially, there are 2 014 red fireflies on the meadow.

1. Prove, that if 5 fireflies remain on the meadow at some time, they don't all have the same colour.
2. Can only 1 firefly remain on the meadow?

Pavel Calábek

A–T–3

Seven distinct positive integers are arbitrarily chosen. Prove that there exist four of them such that their sum is divisible by 4.

Józef Kalinowski

Category B (Individual Competition)

B–I–1

Determine all integers n such that the fraction

$$\frac{8n-1}{11n-2}$$

is reducible.

Stanislav Trávníček

B–I–2

Let x, y be real numbers with $xy = 4$. Determine the smallest possible value of the expression

$$U = |x^3 + x^2y + xy^2 + y^3|.$$

Further, determine all pairs (x, y) of real numbers for which the expression U achieves the smallest value.

Jaroslav Švrček

B–I–3

Prove that the sum of the lengths of two segments connecting the midpoints of opposite sides of an arbitrary quadrilateral is less than the sum of the lengths of its diagonals.

Jacek Uryga

B–I–4

At each vertex of a square exactly one number from the set $\{-1; 1\}$ is written. In one step we change the number written at each vertex to the product of this number and the two adjacent numbers. Prove that we can get numbers 1 at all vertices after some steps, if and only if the initial marking of all vertices of this square is with the four numbers 1.

Jaroslav Švrček

Category B (Team Competition)

B–T–1

In the domain of integers solve the equation

$$x^6 = y^3 + 4\,069.$$

Józef Kalinowski

B–T–2

Let ABC be an acute-angled triangle. Let E be the perpendicular foot of A on BC and F the perpendicular foot of B on AC. Furthermore, let M be the midpoint of BE and N the midpoint of AF. Prove that the line passing through M perpendicular to AC and the line passing through N perpendicular to BC intersect in the midpoint of EF.

Robert Geretschläger

B–T–3

Find all real solutions of the following system of equations

$$\begin{aligned} 2a^2 &= b^2 - \frac{1}{c^2} + 2, \\ 2b^2 &= c^2 - \frac{1}{a^2} + 2, \\ 2c^2 &= a^2 - \frac{1}{b^2} + 2. \end{aligned}$$

Jaroslav Švrček

Category C (Individual Competition)

C–I–1

Determine the smallest positive integer that can be expressed as the sum of two perfect squares in two ways.

(Note that $10 = 3^2 + 1^2$ is one way of expressing the number 10 as the sum of two perfect squares.)

Robert Geretschläger

C–I–2

We are given a regular hexagon $ABCDEF$ with the area P in the plane. The lines CD and EF intersect at the point G. Determine areas of the triangles ABG and BCG in terms of P.

Józef Kalinowski, Jaroslav Švrček

C–I–3

We are given positive integers m and n such that $m^5 + n^3 = 7901$. Determine the value of $m^3 + n^5$.

Robert Geretschläger

C–I–4

Let $A = 2x83$, $B = 19y6$, $C = 29x6$, $D = 1y54$ be four-digit numbers. Determine all pairs (x, y) of decimal digits such that both numbers $A + B$ and $C - D$ are divisible by 3.

Stanislav Trávníček

Category C (Team Competition)

C–T–1

We are given a 4×4 table consisting of 16 unit squares. Determine the number of ways in which the given table can be covered with 5 congruent straight triominos (rectangles 3×1) such that exactly one unit square in the table remains empty.

Jaroslav Švrček

C–T–2

Two circles $k_1(M_1; r_1)$ and $k_2(M_2; r_2)$ intersect in points S and T. The line M_1M_2 intersects k_1 in points A and B and k_2 in C and D such that B lies in the interior of k_2 and C lies in the interior of k_1. Prove that the lines SC and SB trisect the angle ASD if and only if $|\angle M_1SM_2| = 90°$.

Robert Geretschläger

C–T–3

The sum of squares of four (not necessarily different) positive integers a, b, c, d is equal to 100, i.e.

$$a^2 + b^2 + c^2 + d^2 = 100 \quad \text{with} \quad a \geq b \geq c \geq d > 0.$$

a) Determine the largest possible value of $a - d$. Explain why a larger value cannot be possible.
b) Determine the smallest possible value of $a - d$. Explain why a smaller value cannot be possible.

Robert Geretschläger

Solutions

Category A (Individual Competition)

A–I–1

Since both sides of the inequality $|a+c| < |a-c|$ are non-negative real numbers with $a \neq 0$, we equivalently get (after squaring of both sides and easy manipulation) the inequality $ac < 0$ and also $c/a < 0$. This implies that also inequality $b^2 - 4ac > 0$ holds, i.e. the discriminant of the quadratic equation $ax^2 + bx + c = 0$ is a positive real number. Therefore this equation has two real roots x_1 and x_2.

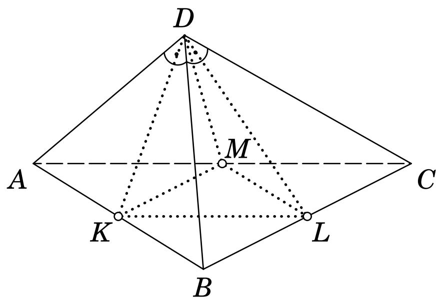
Moreover, applying Vièta's formula we obtain $x_1 x_2 = c/a < 0$, which means that the real roots x_1 and x_2 have opposite signs and the proof is finished.

A–I–2

Since ABD, BCD and CAD are right-angled triangles, then the following equalities

$$|KD| = \tfrac{1}{2}|AB|, \qquad |LD| = \tfrac{1}{2}|BC|, \qquad |MD| = \tfrac{1}{2}|CA|$$

hold.



Since $|KD| = |KB|$ and $|LD| = |LB|$, we can see that the triangles KLD and KLB are congruent (*sss*). Similarly the pairs of triangles LMD, LMC and MKD, MKA are also congruent. Therefore we have

$$|\angle KDL| + |\angle LDM| + |\angle MDK| = |\angle KBL| + |\angle LCM| + |\angle MAK| =$$
$$= |\angle ABC| + |\angle BCA| + |\angle CAB| = 180^\circ,$$

which proves the given statement.

A–I–3

Substituting $y = 0$ in the initial equation we obtain

$$f(f(x)) = x - f(0) \tag{1}$$

and thus

$$f(f(x)) - x = -f(0) \quad \text{for all } x \in \mathbb{R}. \tag{2}$$

If we put $x = f(y)$ into the given equation, then we get

$$f(f(f(y)) - y) = 0. \tag{3}$$

Using (2) in (3) we get $f(-f(0)) = 0$. Further, if we put $x = -f(0)$ into (1), we obtain

$$f(f(-f(0))) = -f(0) - f(0) = -2f(0).$$

The last equation yields $f(0) = -2f(0)$ and so $f(0) = 0$. Finally, putting $x = 0$ into the initial equation we get

$$f(-y) = f(f(0) - y) = -f(y) \quad \text{for all} \quad y \in \mathbb{R}.$$

This means that f is an odd function and the proof is complete.

A–I–4

For $n = 1$ we have $a_1 = 1$.

For $n = 2$ we will prove that no such positive integers exist. If there exist two different positive integers $a_1 < a_2$ such that

$$\frac{1}{a_1} + \frac{1}{a_2} = 1, \qquad \text{we have} \qquad \frac{1}{a_1} > \frac{1}{2}, \qquad \text{i.e.} \qquad a_1 < 2$$

which implies $a_1 = 1$. This is a contradiction with the assumption of existence of such a_1 and a_2.

For $n = 3$ we can see

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6},$$

so $a_1 = 2$, $a_2 = 3$ and $a_3 = 6$. Now we can use mathematical induction.

Using mathematical induction we further show that for each positive integer $n \geq 3$ there exist positive integers $a_1 < a_2 < \ldots < a_n$ such that

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}.$$

For $n = 3$ it is proven above and next we use the following identity

$$\frac{1}{a_n} = \frac{1}{a_n + 1} + \frac{1}{a_n(a_n + 1)}.$$

Since $a_n < a_n + 1 < a_n(a_n + 1)$ this identity implies the existence of an $(n + 1)$-tuple of mutually distinct positive integers such that

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{n-1}} + \frac{1}{a_n + 1} + \frac{1}{a_n(a_n + 1)}.$$

For example

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{42 \cdot 43} = \ldots$$

Conclusion. There exist mutually distinct positive integers a_1, $a_2, \ldots, a_n$ such that

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}$$

only for $n = 1$ and for each $n \geq 3$.

Category A (Team Competition)

A–T–1

We first note that applying the Cauchy-Schwarz inequality to the vectors

$$\left(\frac{u}{\sqrt{x}}, \frac{v}{\sqrt{y}}, \frac{w}{\sqrt{z}}\right) \qquad \text{and} \qquad (\sqrt{x}, \sqrt{y}, \sqrt{z})$$

yields

$$(u + v + w)^2 \leq \left(\frac{u^2}{x} + \frac{v^2}{y} + \frac{w^2}{z}\right)(x + y + z)$$

or

$$\frac{(u+v+w)^2}{x+y+z} \le \frac{u^2}{x} + \frac{v^2}{y} + \frac{w^2}{z}.$$

Setting $u = a$, $v = 1$, $w = b$, $x = 1$, $y = a$ and $z = 8$ we obtain

$$\frac{(a+1+b)^2}{1+a+8} \le a^2 + \frac{1}{a} + \frac{b^2}{8}$$

which is equivalent to the given inequality

$$(a+9)\left(a^2 + \frac{1}{a} + \frac{b^2}{8}\right) \ge (a+b+1)^2.$$

Equality holds iff the vectors are multiples of each other, i.e. if

$$\frac{u}{x} = \frac{v}{y} = \frac{w}{z} \quad \text{or} \quad a = \frac{b}{8} = \frac{1}{a}$$

holds. Since $a = \frac{1}{a}$ and $a > 0$ hold, we have $a = 1$ and therefore $b = 8$ as the only case in which equality holds.

A–T–2

Let us consider the meadow with r red, g green and b blue fireflies. Let I be the remainder of the term $r + 2g + 3b$ after division by the number 5. By direct verification we can show that I is invariant. Initially, we have $r = 2014$, $b = g = 0$, and thus $I = 4$.

1. If 5 fireflies of the same colour remain on the meadow, we have $I = 0$, which contradict $I = 4$.
2. If 1 firefly remains on the meadow, we have $I = 1$ for the red firefly, $I = 2$ for the green one and $I = 3$ for the blue firefly. All cases contradicts $I = 4$. So only 1 firefly can not remain on the meadow.

A–T–3

Each of the seven arbitrarily chosen numbers has, after division by 4, a remainder from the set $\{0, 1, 2, 3\}$.

a) If at least five of the chosen numbers have remainders 0 or 2, then we consider 2 cases.

   - ▷ If among them there are at least four of these five numbers have the same remainder, then we choose these four of them and their sum is divisible by 4.
   - ▷ Otherwise there are at least two of these numbers with the remainder 0 and simultaneously at least two of them with remainder 2. In this subcase we take two numbers with the remainder 0 and two numbers with the remainder 2. Altogether, the sum of these four numbers is divisible by 4.

b) If (exactly) four of the seven chosen numbers have remainder 0 or 2, then we will consider the following cases.

   - ▷ If all of them have the same remainder, we take these four numbers and their sum is divisible by 4.
   - ▷ Otherwise there is at least one number with the remainder 2 and at least one with remainder 0. Further let us focus on three other numbers with odd remainders. At least two of them have the same remainder. In this case we take two of these one number with the remainder 0 and one numbers with remainder 2. Such four numbers obviously have a sum which is divisible by 4.

c) If at most three of the seven chosen numbers have remainders 0 or 2, we add 1 to all seven numbers. Now by a) or b) we can choose four of them with their sum divisible by 4. If we now subtract 1 from each oh these numbers, their sum remains divisible by 4.

Remark. It is easy to check, that this statement is generally not valid for six chosen numbers. For example, we can consider arbitrary numbers with the remainders 0, 0, 0, 1, 1, 1, respectively.

Category B (Individual Competition)

B–I–1

It is easy to see that the denominator of the given fraction is non-zero for all integer values of n.

Let D be the greatest common divisor of two integers $8n - 1$ and $11n - 2$. Then D also divides the number $11(8n - 1) - 8(11n - 2) = 5$. This implies $D \in \{1; 5\}$. The given fraction is therefore reducible if and only if the number 5 divides both integers $8n - 1$ and $11n - 2$.

Since $8n - 1 = 8(n - 2) + 15$ we have $n - 2 = 5k$ for arbitrary integers k, i.e. $n = 5k + 2$. We can easily check that for such n the denominator $11n - 2$ of the given fraction is divisible by 5, as well.

Conclusion. The given fraction is reducible (by the number 5) for any $n = 5k + 2$ with arbitrary integer k.

B–I–2

Firstly, we can see that both numbers x, y are of the same sign, i.e. either both are positive or both are negative numbers. Moreover, for advanced values of numbers x, y the value of U is the same for each of the pairs (x, y) and $(-x, -y)$. We can therefore solve the given problem for $x > 0$, $y > 0$. We then have

$$U = x^3 + x^2y + xy^2 + y^3 = x^2(x + y) + y^2(x + y) = (x + y)(x^2 + y^2).$$

We can estimate each factor on the right-hand side of the last equation using the well-known AM-GM inequality in the form $a + b \geq 2\sqrt{ab}$, which is true for arbitrary positive real numbers a, b. This implies

$$U = (x + y)(x^2 + y^2) \geq (2\sqrt{xy}) \cdot (2xy) = 4 \cdot 8 = 32,$$

in which the equality holds if and only if $x = y = 2$.

Conclusion. The given expression achieves the smallest value $U = 32$ exactly for two pairs of real numbers (x, y), more precisely for $(x, y) = (2; 2)$ and for $(x, y) = (-2; -2)$.

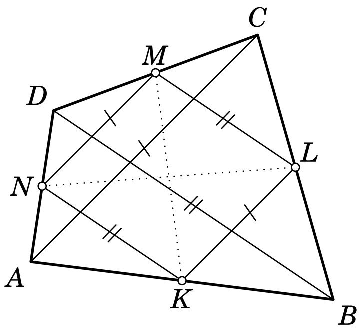
B–I–3

Let us consider an arbitrary quadrilateral $ABCD$. Further, let K, L, M, N be midpoints of its sides AB, BC, CD, DA, respectively. Since KL, LM, MN, NK are co-called *midlines* in the triangles ABC, BCD, CDA, DAB, respectively, we can see that the following assertions

$$KL \parallel CA \parallel MN, \qquad LM \parallel BD \parallel NK$$

hold. Moreover we have

$$|KL| = |MN| = \tfrac{1}{2}\,|AC| \qquad \text{and} \qquad |LM = |NK| = \tfrac{1}{2}\,|BD|. \tag{4}$$



This implies, that $KLMN$ is a parallelogram (the *Varignon's* parallelogram). Using the well-known triangle inequality for diagonals of this parallelogram we finally get

$$|KM| < |KL| + |LM| \qquad \text{and} \qquad |LN| < |LM| + |MN|.$$

Summing up last two inequalities and according to (4), we obtain

$$|KM| + |LN| < (|KL| + |MN|) + 2\,|LM| = |AC| + |BD|,$$

which completes the proof.

Remark. It is obvious, that the initial statement is true for convex and also for non-convex given quadrilaterals.

B–I–4

Let us consider a square $A_1A_2A_3A_4$. Let a_i be numbers of the set $\{-1;1\}$ written at the vertices A_i ($i = 1,2,3,4$) in some step. Let us denote further $k = a_1a_2a_3a_4$. In the following step we therefore obtain at the vertices A_i values ka_{i+2} ($a_5 = a_1$, $a_6 = a_2$). Moreover, in each step we get the same product $k^5 = k$ of all four numbers.

Therefore, if we get numbers 1 at all vertices in some step, then it must be $a_1 = a_2 = a_3 = a_4$ in the previous step. Therefore we get $k = 1$, which follows $a_1 = a_2 = a_3 = a_4 = 1$. Thus, by this way we get numbers 1 at all vertices initially. On the other hand in this case we can see that the numbers at all vertices are 1 in all steps.

Category B (Team Competition)

B–T–1

We can rewrite the given equation in the equivalent form

$$(x^2 - y)(x^4 + x^2y + y^2) = 4069.$$

Since $x^4+x^2y+y^2 = (x^2+\frac{1}{2}y)^2+\frac{3}{4}y^2 \geq 0$, both factors x^2-y and $x^4+x^2y+y^2$ must be positive.

First, we can see that for each $y \in \{-2,-1,0\}$ no integer x fulfilling the given equation exists. For such y we have $x^6 \in \{4061, 4068, 4069\}$, for which no integer x exists.

We further show that for any other integer y the inequality

$$x^2 - y < x^4 + x^2y + y^2$$

holds. Rewriting this inequality we obtain

$$x^4 + (y-1)x^2 + y^2 + y > 0$$

with unknown x. For the variable x^2 and arbitrary integer parameter $y \notin \{-2,-1,0\}$ this is a quadratic polynomial with negative discriminant Δ, because

$$\Delta = (y-1)^2 - 4(y^2 + y) = -3y^2 - 6y + 1 = 4 - 3(y+1)^2 < 0.$$

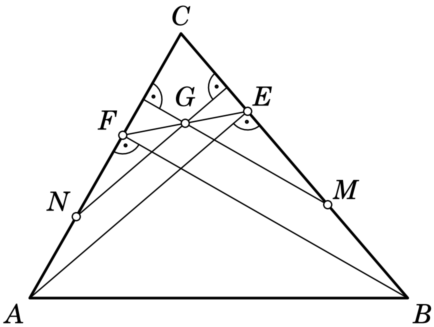
Since $0 < x^2 - y < x^4 + x^2y + y^2$ and 4069 is the product of two primes 13 and 313 we can further consider only two possible cases

1. $x^2 - y = 1$ and $x^4 + xy + y^2 = 4069$. Substituting $x^2 = y + 1$ we obtain the equation $y^2 + y - 1356 = 0$ with the discriminant $\Delta = 5\sqrt{217}$ and y isn't an integer.
2. $x^2 - y = 13$ and $x^4 + xy + y^2 = 313$. Substituting $x^2 = y + 13$ we obtain the equation $y^2 + 13y - 48 = 0$ with roots $y_1 = -16$ and $y_2 = 3$. For $y_1 = -16$ we have $x^2 = -3$ which is impossible. For $y_2 = 3$ we have $x^2 = 16$ and so $x_1 = -4$ and $x_2 = 4$.

Conclusion. The given equation has two integer solutions, namely $(x, y) \in \{(-4; 3), (4; 3)\}$.

B–T–2

Since M is the midpoint of the segment BE, a perpendicular to AC passing through M (parallel with BF) is the midline of the triangle BEF. Thus this line also passes through the midpoint G of the segment EF. Similarly, the line passing through N parallel to AE passes through the midpoint G, as well. The proof is finished.



Another solution. The homothety with center in E and ratio $\frac{1}{2}$ maps B to M and the altitude of the triangle through B to the line perpendicular to AC through M. Since this homothety also maps F to the midpoint of EF, we see that the line through M and perpendicular to AC certainly passes through the midpoint of EF. By completely analogous reasoning with the center of homothety in F and mapping the altitude in A to the line perpendicular to BC and passing through N, we see that the two perpendicular lines both pass through the mid-point of EF, as claimed.

B–T–3

Adding up all three equations, a short calculation gives us

$$\left(a^2+\frac{1}{a^2}\right)+\left(b^2+\frac{1}{b^2}\right)+\left(c^2+\frac{1}{c^2}\right)=6.$$

Since for all non-zero real numbers a, b, c the inequalities

$$a^2+\frac{1}{a^2}\geq 2, \qquad b^2+\frac{1}{b^2}\geq 2, \qquad c^2+\frac{1}{c^2}\geq 2,$$

hold, we have

$$a^2+\frac{1}{a^2}=b^2+\frac{1}{b^2}=c^2+\frac{1}{c^2}=2$$

and therefore $(a, b, c) = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ with $\varepsilon_i \in \{-1; 1\}$ for $i = 1, 2, 3$.

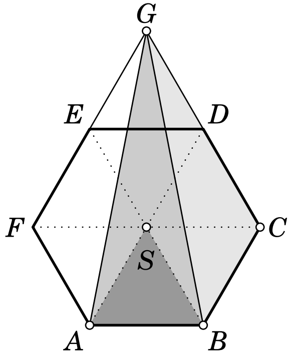
Conclusion. The given system of equations has exactly eight solutions, namely $(a, b, c) = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ with $\varepsilon_i \in \{-1; 1\}$, $i = 1, 2, 3$.

Remark. Solution is complete once we have checked the validity for each of these pairs.

Category C (Individual Competition)

C–I–1

The smallest such number is $50 = 5^2 + 5^2 = 1^2 + 7^2$. (Note that the requirement that the two perfect squares must be different yields the solution $65 = 4^2 + 7^2 = 8^2 + 1^2$.)



C–I–2

Let P be the area of regular hexagon $ABCDEF$ and Q be the area of equilateral triangle EDG. From the picture it is easy to see, that the area of the equilateral triangle DEG is six-times smaller than the area P of the given regular hexagon $ABCDEF$ which is composed of from six smaller (and with DEG) congruent equilateral triangles of the same area Q, therefore $Q = P/6$. Further we can see that the area of the pentagon $ABCGE$ is equal to $P + Q = P + \frac{1}{6}P = \frac{7}{6}P$. Let S denote the center of the regular hexagon $ABCDEF$. Then the area R of the isosceles triangle ABG is three-times larger than the area of the equilateral triangle ABS, because the altitude from G to the side AB of the triangle ABG is three-times longer than the altitude from S to the same side AB in the equilateral triangle ABS. Therefore we finally have

$$R = 3Q = 3 \cdot \frac{P}{6} = \frac{P}{2}.$$

Thus for the area T of the triangle BCG (according to the symmetry of the pentagon $ABCGF$ with respect to perpendicular bisector of its side AB) $2T + R = 7Q$ holds. This implies

$$T = \frac{7Q - R}{2} = \frac{1}{2}\left(\frac{7P}{6} - \frac{P}{2}\right) = \frac{P}{3}.$$

Conclusion. The triangles ABG and BCG have areas $\frac{1}{2}P$ and $\frac{1}{3}P$, respectively.

Remark. The area T of the triangle BCG should be computed similarly to the area of ABG (the altitude from G to the side BC in the triangle BCG is twice of the altitude from S to the side BC in the equilateral triangle BCS).

C–I–3

Noting that the fifth powers of integers are

$$1, 32, 243, 1024, 3125, 7776, \ldots,$$

we see that $m \le 6$ must hold. Since $7901 - 7776 = 125 = 5^3$, we see that $m = 6$ and $n = 5$ holds. (A quick check shows that subtracting any other number of the list from 7901 does not yield a perfect cube.) We therefore have $m^3 + n^5 = 6^3 + 5^5 = 3341$.

C–I–4

We have

$$\begin{aligned} A + B &= 3 \cdot 1000 + (x + 9) \cdot 100 + (8 + y) \cdot 10 + 9 \\ &= 3\,(1330 + 33x + 3y) + x + y - 1, \\ C - D &= 1 \cdot 1000 + (9 - y) \cdot 100 + (x - 5) \cdot 10 + 2 \\ &= 3\,(617 + 3x - 33y) + x - y + 1. \end{aligned}$$

We can see that

$$3 \mid (A + B) \qquad \text{iff} \qquad 3 \mid (x + y - 1)$$

and similarly

$$3 \mid (C - D) \qquad \text{iff} \qquad 3 \mid (x - y + 1).$$

These two conditions are equivalent to

$$\begin{aligned} &3 \mid [(x + y - 1) + (x - y + 1)] = 2x, \\ &3 \mid [(x + y - 1) - (x - y + 1)] = 2(y - 1) \end{aligned}$$

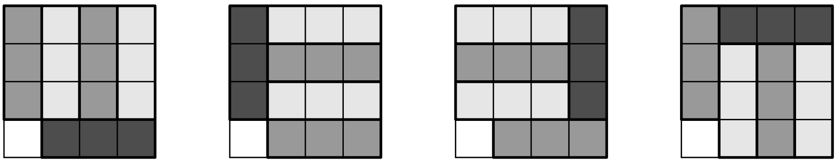
and it follows $3 \mid x$ and $3 \mid (y - 1)$. Since x and y are decimal digits we have $x \in \{0, 3, 6, 9\}$ and $y \in \{1, 4, 7\}$.

Conclusion. The given problem has a total of 12 solutions (x, y) with $x \in \{0, 3, 6, 9\}$ and $y \in \{1, 4, 7\}$.

Category C (Team Competition)

C–T–1

We can consider two possibilities for a position of the empty cell by covering the given 4×4 table by five straight triominos. Since the square table is symmetric with respect to its center, we will consider at first only one of four cases of a required covering with an empty cell at one of corresponding vertices of the square 4×4 table. For the cell at bottom left vertex we get the four possibilities in the picture.



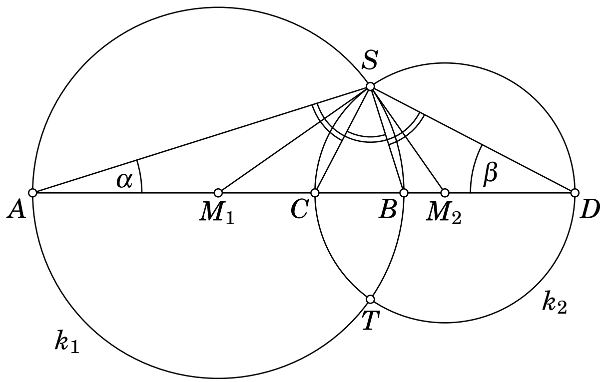
Further, it is easy to see that any other cases of the empty cell square don't fulfill the conditions of the given problem.

Conclusion. There exist $4 \cdot 4 = 16$ possibilities for the covering of the given square table under the conditions of the problem.

C–T–2

Naming $|\angle SAB| = \alpha$ and $|\angle SDC| = \beta$, we have $|\angle SBA| = 90^\circ - \alpha$, and therefore $|\angle BSD| = 90^\circ - \alpha - \beta$. On the other hand, we also have $|\angle SCD| = 90^\circ - \beta$, and therefore $|\angle CSA| = 90^\circ - \alpha - \beta$, and these two angles are therefore certainly equal. Since $|\angle ASB| = 180^\circ - \alpha - \beta$, we have $|\angle CSB| = \alpha + \beta$.

We now consider the case in which SC and SB trisect $\angle ASD$. This is equivalent to $90^\circ - \alpha - \beta = \alpha + \beta$, which is itself equivalent to $\alpha + \beta = 45^\circ$. Since $|\angle SM_1B| = 2\alpha$ and $|\angle SM_2C| = 2\beta$, this is equivalent to $|\angle SM_1B| + |\angle SM_2C| = |\angle SM_1M_2| + |\angle SM_2M_1| = 2(\alpha + \beta) = 90^\circ$, and since $|\angle M_1SM_2| = 180^\circ - (|\angle SM_1M_2| + |\angle SM_2M_1|)$, this is equivalent to $|\angle M_1SM_2| = 90^\circ$ as claimed.



C–T–3

a) We first note that since $10^2 = 100$, we must have $a \leq 9$. Furthermore, we certainly have $d \geq 1$. Since we can write

$$100 = 9^2 + 3^2 + 3^2 + 1^2,$$

we can have $a = 9$ and $d = 1$, and therefore $a - d = 8$, which is therefore certainly the largest possible value.

b) Since $a \geq d$ i.e. $a - d \geq 0$, the smallest possible value of $a - d$ is 0. Since

$$100 = 5^2 + 5^2 + 5^2 + 5^2,$$

we can have $a = d = 5$ and therefore $a - d = 0$, and this is certainly the smallest possible value.

Results

Category A (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Aleksandar Nezabravkov Cherganski	SMG Sofia	8	8	8	8	**32**
	Alexander Antonov Tenev	SMG Sofia	8	8	8	8	**32**
	Denitsa Hristova Markova	SMG Sofia	8	8	8	8	**32**
	Petr Vincena	GJŠ Přerov	8	8	8	8	**32**
5.	Marian Poljak	GJŠ Přerov	5	8	8	8	**29**
6.	Tsvetomir Mihajlov Mihajlov	SMG Sofia	3	8	8	8	**27**
7.	Markéta Calábková	GJŠ Přerov	8	8	3	6	**25**
8.	Viet Anh Nguyen	BRG Graz	8	0	0	8	**16**
9.	Markus Ruprechter	BRG Graz	7	1	0	6	**14**
10.	Heinz Prach	BRG Graz	5	0	0	8	**13**
	Tomáš Kremel	GJŠ Přerov	4	8	1	0	**13**
12.	Gerda Prach	BRG Graz	7	1	0	4	**12**
13.	Jakub Paliga	I. LO Chorzów	8	1	2	0	**11**
14.	Janek Lukasz	I. LO Chorzów	8	0	0	0	**8**
15.	Jan Gocník	GJŠ Přerov	4	1	2	0	**7**
	Jan Šarman	GMK Bílovec	6	0	1	0	**7**
	Michal Ślusarczyk	I. LO Chorzów	6	1	0	0	**7**
18.	Jan Krejčí	GMK Bílovec	6	0	0	0	**6**
19.	Tomáš Moravec	GMK Bílovec	1	1	0	0	**2**
20.	Svatopluk Rusek	GMK Bílovec	0	0	0	0	**0**

Category B (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Vasil Radkov Yordanov	SMG Sofia	8	8	8	8	**32**
2.	Svetoslav Stefanov Radoev	SMG Sofia	8	7	8	8	**31**
3.	Georgi Toshkov Dimitrov	SMG Sofia	8	8	7	4	**27**
4.	Teodor Dimitrov Aleksiev	SMG Sofia	2	3	5	8	**18**
5.	Konstantin Andritsch	BRG Graz	0	6	0	8	**14**
6.	Benedikt Andritsch	BRG Graz	4	1	2	6	**13**
7.	Martin Kubeša	GJŠ Přerov	0	4	0	8	**12**
	Sebastian Borówka	I. LO Chorzów	2	2	0	8	**12**
9.	Artur Wyciślok	I. LO Chorzów	2	1	0	8	**11**
	Daniel Horiatakis	BRG Graz	1	2	0	8	**11**
11.	Jan Šuta	GJŠ Přerov	0	1	0	8	**9**
	Tereza Špalková	GJŠ Přerov	0	1	0	8	**9**
13.	Petra Nyklová	GMK Bílovec	1	0	0	6	**7**
14.	Denisa Chytilová	GJŠ Přerov	2	2	0	2	**6**
	Zuzana Beigerová	GMK Bílovec	0	0	0	6	**6**
16.	Jiří Grygar	GMK Bílovec	1	3	0	1	**5**
17.	Doris Prach	BRG Graz	0	0	0	4	**4**
18.	Krzysztof Kleist	I. LO Chorzów	0	0	0	3	**3**
19.	Damian Wa?oszek	GMK Bílovec	2	0	0	0	**2**
20.	Karol Pieczka	I. LO Chorzów	0	1	0	0	**1**

Category C (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Boris Aleksandrov Barbov	SMG Sofia	8	8	8	8	**32**
	Ivo Vladislavov Petrov	SMG Sofia	8	8	8	8	**32**
3.	Joana Nikolaeva Nikolova	SMG Sofia	5	8	8	8	**29**
4.	Michaela Svatošová	GMK Bílovec	8	8	8	1	**25**
5.	Julie Dostalíková	GJŠ Přerov	8	0	8	8	**24**
6.	Tomáš Křižák	GMK Bílovec	1	8	8	4	**21**
7.	Kostadin Tsvetomirov Penchev	SMG Sofia	8	4	0	8	**20**
8.	Katarzyna Kępińska	I. LO Chorzów	1	8	0	8	**17**
9.	Sebastian Fellner	BRG Graz	0	8	0	8	**16**
10.	Berenika Čermáková	GMK Bílovec	5	0	6	1	**12**
	Lukáš Kremel	GJŠ Přerov	0	0	4	8	**12**
12.	Bára Tížková	GMK Bílovec	1	4	0	6	**11**
	Tomasz Maroszczuk	I. LO Chorzów	2	8	0	1	**11**
14.	Grzegorz Pielot	I. LO Chorzów	0	8	0	2	**10**
	Karolina Vellechová	GJŠ Přerov	2	7	0	1	**10**
	Zdeněk Kroča	GJŠ Přerov	8	1	0	1	**10**
17.	Kirolos Louies	BRG Graz	5	4	0	0	**9**
18.	Anna Wiśniewska	I. LO Chorzów	2	0	0	0	**2**
	Manuel Hasenbichler	BRG Graz	1	0	0	1	**2**

Category A (Team Competition)

Rank	School	1	2	3	∑
1.	GJŠ Přerov	8	8	8	**24**
2.	SMG, Sofia	8	8	7	**23**
3.	BRG Kepler, Graz	7	0	8	**15**
4.	GMK Bílovec	8	0	5	**13**
5.	I. LO Chorzów	0	0	1	**1**

Category B (Team Competition)

Rank	School	1	2	3	∑
1.	SMG, Sofia	7	8	7	**22**
2.	I. LO Chorzów	3	6	6	**15**
3.	BRG Kepler, Graz	2	8	2	**12**
4.	GJŠ Přerov	2	0	2	**4**
5.	GMK Bílovec	1	0	2	**3**

Category C (Team Competition)

Rank	School	1	2	3	∑
1.	SMG, Sofia	8	4	8	**20**
2.–3.	I. LO Chorzów	6	0	8	**14**
2.–3.	GMK Bílovec	6	0	8	**14**
3.–4.	GJŠ Přerov	8	0	4	**12**
3.–4.	BRG Kepler, Graz	8	0	4	**12**

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