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MATHEMATICAL DUEL '16

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Preface

The 24th Mathematical Duel was held in Ostrava from 12th till 16th March 2016. It was the second of three competitions planned as part of the Duel Plus project, which is entirely financed by the Erasmus Plus programme. The students from three age groups – Category A (age 17–19), Category B (age 15–17), Category C (age 13–15) – took part in individual and team competition. The teams came from Austrian, Polish and Czech schools, namely Bundesrealgymnasium Kepler in Graz, Akademicki Zespół Szkół Ogólnokształcących in Chorzów, Gymnázium Jakuba Škody in Přerov and Gymnázium Mikuláše Koperníka in Bílovec.

The individual competition started on 13th March at 9 a.m. The students in each age group were allowed two hours and thirty minutes to solve four problems (they could achieve maximum score of 8 points for each problem). Since all the problems were formulated in English, the teachers in charge of their teams helped them understand the content. The students could provide solutions either in their native languages or in English.

The second part of the Duel, i.e. team competition started after coffee and snack break. The participating schools were allowed to send 12 persons – four persons for each age group. Thus, four students representing the same school and age group could join one team. The teams were told to solve three problems within 100 minutes.

This booklet contains all problems with solutions and results of the 24th Mathematical Duel from the year 2016.

The authors

Problems

Category A (Individual Competition)

A–I–1

Find the largest positive integer n with the following property: The product

$$(k+1)\cdot(k+2)\cdot(k+3)\cdot\ldots\cdot(k+2016)$$

is divisible by 2016^n for every positive integer k.

Jaroslav Švrček

A–I–2

In the domain of real numbers, solve the system of equations

$$\begin{aligned} x &= p^2 + y^2, \\ y &= q^2 + z^2, \\ z &= r^2 + x^2 \end{aligned}$$

with non-negative real parameters p, q, r satisfying $p + q + r = \frac{3}{2}$.

Józef Kalinowski

A–I–3

We are given a square $ABCD$ in the plane. Find the locus of vertices P of all right-angled isosceles triangles APQ with the right angle at P such that the vertex Q lies on the side CD of the given square.

Jaroslav Švrček

A–I–4

On every square of a 10×10 table sits exactly one flea. At a signal all fleas jump diagonally over one square onto another square of the table. Then, there are some squares with several fleas and some empty squares. Find the minimum possible number of empty squares.

Pavel Calábek

Category A (Team Competition)

A–T–1

Prove that for every positive integer n there exists an integer m divisible by 5^n which consists exclusively of odd digits.

Jacek Uryga

A–T–2

A cube F with edge length 9 is divided into 9^3 small cubes with edges of the length 1 by planes parallel to its faces. One small cube is removed from the center of each of the six faces of the cube F. Let us denote the resulting solid by G. Is it possible to build the solid G only from rectangular cuboids with the dimensions $1 \times 1 \times 3$?

Jacek Uryga

A–T–3

Determine all polynomials $P(x)$ with real coefficients such that for each real number x the equation

$$P(x^2) - 3x^3 + 15x^2 - 24x + 12 = P(x)\,P(2x)$$

holds.

Pavel Calábek

Category B (Individual Competition)

B–I–1

Let a, b, c be arbitrary real numbers. Prove that the inequality

$$a^2 + 5b^2 + 4c^2 \geq 4(ab + bc)$$

holds. When does equality hold?

Robert Geretschläger

B–I–2

Determine in how many ways one can assign numbers of the set $\{1, 2, \ldots, 8\}$ to the vertices of a cube $ABCDEFGH$ such that the sum of any two numbers at vertices with a common edge is an odd number.

Jaroslav Švrček

B–I–3

A circle meets each side of a rectangle at two points. Intersection points lying on opposite sides are vertices of two trapezoids. Prove that points of intersection of these two trapezoids lying inside the rectangle are vertices of a cyclic quadrilateral.

Jacek Uryga

B–I–4

For how many numbers from the set $\{1, 2, 3, \ldots, 2016\}$ is a remainder of its square after division by 2016 equal to 1?

Pavel Calábek

Category B (Team Competition)

B–T–1

Determine all 4-digit palindromic numbers n (i.e. a number that reads the same from front to back and from back to front), such that $17n$ is a perfect square.

Robert Geretschläger

B–T–2

A right-angled triangle ABC in the plane is given. Its legs BC and AC are hypotenuses of two right-angled isosceles triangles BCP and ACQ erected outside ABC. Let D be the vertex of the right-angled isosceles triangle ABD with hypotenuse AB erected inside ABC. Prove that the point D belongs to the line PQ.

Jaroslav Švrček

B–T–3

A set $\mathcal{A}$ consists exclusively of positive integers not divisible by 7. How many elements must the set $\mathcal{A}$ at least contain in order to be sure that there is an non-empty subset of $\mathcal{A}$, such that the sum of the squares of the elements of this subset is divisible by 7?

Jacek Uryga

Category C (Individual Competition)

C–I–1

The inequality $a^2 + 2b^2 \geq pab$ is known to hold for all real values of a and b.

a) Determine any value of p, such that this is true and prove why this is true.
b) Does this inequality hold for all real numbers a, b, provided $p = 2$ and $p = 3$? Why?

Robert Geretschläger, Jaroslav Švrček

C–I–2

Each vertex of a regular hexagon $ABCDEF$ is coloured by one of three colours (red, white and blue) such that each colour is used exactly twice. Determine in how many ways we can do this, when any two adjacent vertices of the hexagon are coloured by distinct colours.

Jaroslav Švrček

C–I–3

We are given an equilateral triangle ABC in the plane. Let D be an interior point of the side AC. On the ray BC, beyond the point C, lies the point E such that $|AD| = |CE|$. Prove that $|BD| = |DE|$.

Józef Kalinowski

C–I–4

We are given two positive real numbers x and y. Their arithmetic mean $A = (x+y)/2$ and their geometric mean $G = \sqrt{xy}$ are in the ratio $A : G = 5 : 4$. Determine the ratio $x : y$.

Robert Geretschläger

Category C (Team Competition)

C–T–1

Let a, b, c be arbitrary non-zero real numbers. Let us denote

$$A = \frac{a^2 + b^2}{c^2}, \quad B = \frac{b^2 + c^2}{a^2}, \quad C = \frac{c^2 + a^2}{b^2}$$

and further $P = A \cdot B \cdot C$ and $S = A + B + C$. Determine all possible values of the difference $P - S$.

Józef Kalinowski

C–T–2

We are given two circles c_1 and c_2 with midpoints M_1 and M_2, respectively. The radius of c_1 is r and the radius of c_2 is $2r$. Furthermore, we are given that $|M_1M_2| = r\sqrt{2}$. Let P be a point on c_2 with the property that the tangents from P to c_1 are mutually perpendicular. Determine the area of the triangle M_1M_2P in terms of r.

Robert Geretschläger

C–T–3

We are given the interesting number 5040. Like some other numbers, this number can be expressed as the product of several consecutive integers. (For example, the number 1320 can be expressed in the form $1320 = 10 \cdot 11 \cdot 12$ as the product of three consecutive integers.)

a) Express 5040 as the product of 4 consecutive integers.
b) Express 5040 as the product of 6 consecutive integers.
c) Prove that 5040 cannot be expressed as the product of two consecutive integers.

Robert Geretschläger

Solutions

Category A (Individual Competition)

A–I–1

We can write the number 2 016 in the canonical form (using primes only) in the form $2\,016 = 2^5 \cdot 3^2 \cdot 7$. It is easy to see that every second factor in the product

$$(k+1)\cdot(k+2)\cdot(k+3)\cdot\ldots\cdot(k+2016)$$

is divisible by 2, similarly every third factor of this product is divisible by 3 and every seventh factor is divisible by 7.

We find the largest positive integer n_7 such that 7^{n_7} is a divisor of the given product for every positive integer k. Since every seventh factor of the given product is divisible by 7, we have $2016/7 = 288$ numbers divisible at least by 7. Moreover, every 49th factor is divisible by $49 = 7^2$. Thus, we altogether have $\lfloor 2016/49 \rfloor = 41$ such numbers ($\lfloor x \rfloor$ is the lower integer part of a real number x). Finally, every 343rd factor is divisible by $343 = 7^3$, i.e. further we have in this product $\lfloor 2016/343 \rfloor = 5$ such numbers (and $7^4 > 2016$). It follows $n_7 = 288 + 41 + 5 = 334$. Similarly we find $n_3 = 1004$ and $n_2 = 2010$.

Conclusion. The largest positive integer n satisfying conditions of the problem is

$$n = \min\left\{\left\lfloor \frac{n_2}{5} \right\rfloor, \left\lfloor \frac{n_3}{2} \right\rfloor, n_7\right\} = \min\{402, 502, 334\} = 334.$$

A–I–2

For any non-negative real parameters p, q, r we have (by the QM-AM inequality)

$$\sqrt{\frac{p^2+q^2+r^2}{3}} \ge \frac{p+q+r}{3} \Leftrightarrow p^2+q^2+r^2 \ge \frac{(p+q+r)^2}{3} \ge \frac{1}{3}\cdot\frac{9}{4} = \frac{3}{4}$$

with equality if and only if $p = q = r = \frac{1}{2}$. Adding all three equations, we obtain

$$0 = x^2+y^2+z^2-(x+y+z)+(p^2+q^2+r^2) \ge x^2+y^2+z^2-(x+y+z)+\tfrac{3}{4} =$$
$$= \left(x-\tfrac{1}{2}\right)^2 + \left(y-\tfrac{1}{2}\right)^2 + \left(z-\tfrac{1}{2}\right)^2 \ge 0$$

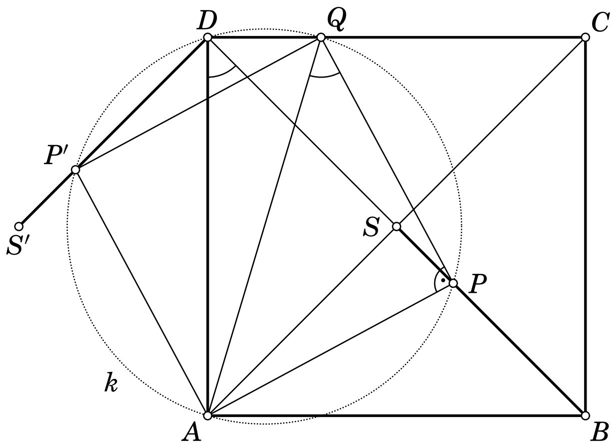
for $p = q = r = \frac{1}{2}$.

After checking we see that the triple $(x,y,z) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the unique solution of the given system of equations.

Conclusion. The system of equations with non-negative real parameters p,q,r such that $p+q+r = \frac{3}{2}$ has the unique solution $x = y = z = \frac{1}{2}$, for $p = q = r = \frac{1}{2}$ only.

A–I–3

Let us consider a right-angled isosceles triangle APQ with the hypotenuse AQ such that the vertex P lies in the same half-plane as the vertex B of the given square with respect to the line AQ. We can see in the picture that the segment AQ is a diameter of the Thales circle k, on which lie the points P and D (ADQ and APQ are both right angles). Thus the points A, P, Q and D lie (in this order) on the same circle (i.e. the quadrilateral $APQD$ is cyclic). This implies that we see the side AP of the quadrilateral $APQD$ from the points Q and D (both points are in the same half-plane with respect to the line AP) under the same angle, namely 45°. Therefore, the point P lies on the segment BS where S is the center of the given square $ABCD$ (its endpoints B, S are simultaneously boundary positions of the points Q on the side CD), because the diagonal BD containing the segment BS holds the angle of the measure 45° with all sides of this square.



Conversely, it is easy to see, that for every point P of the segment BS there exists a point Q of the side CD which is a vertex of a right-angled isosceles triangle APQ with the right angle at P.

Further, let us consider a right-angled isosceles triangle $AP'Q$ with the hypotenuse AQ such that its vertex P' lies in the same half-plane as D with respect to the line AQ. As in the previous case we can easy to see that $AQDP'$ is a cyclic quadrilateral with $|\angle ADP'| = 45^\circ$ and thus the locus of the vertices P' of all right-angled triangles is in this case the segment DS', where S' is the symmetric point to the center S of the given square with respect to the closed segments AD.

Conclusion. The locus of vertices P of all right-angled isosceles triangles with a right angle at P is the pair of closed segments BS and DS'.

A–I–4

Let us label the columns of the table from left to right with 1, 2, 3, 4, 1, 2, 3, 4, 1, 2. There are 30 squares in each of the columns 1, 2 and 20 squares in each of the columns 3, 4. The fleas from the columns 1 jump to the columns 3 and from the columns 3 to the columns 1. Therefore there are at least 10 empty squares in the columns 1 after the signal. Similarly there are at least 10 empty squares in the columns 2. Altogether there are at least 20 empty squares.

1	**2**	**3**	**4**	**1**	**2**	**3**	**4**	**1**	**2**
8	9	10	11	12	13	14	15		
7	22	23	24	25	26	27	28		
6	21	8	9	10	11	12	13	14	15
5	20	7	22	23	24	25	26	27	28
4	19	6	21	32	33	34	35		
3	18	5	20	31	38	39	40		
2	17	4	19	30	37	32	33	34	35
1	16	3	18	29	36	31	38	39	40
		2	17			30	37		
		1	16			29	36		

On the other hand, if (in the previous picture) the fleas from the squares with the same number exchange their positions and fleas from the empty squares jump to some suitable square, we can see, that there remains exactly 20 empty squares.

Conclusion. The minimum possible number of empty squares is therefore 20.

Category A (Team Competition)

A–T–1

We prove a stronger assumption: for every positive integer n there exists an integer m divisible by 5^n which consists exclusively of n odd digits.

Note first that for every positive integer n and q the numbers $2^n + q$, $3 \cdot 2^n + q$, $5 \cdot 2^n + q$, $7 \cdot 2^n + q$, $9 \cdot 2^n + q$ give five different remainders when divided by 5. Indeed, if two of the remainders were equal, then the difference of those numbers would be divisible by 5, which is not possible.

For $n = 1$ we can choose $m = 5$. Now suppose, for some n we have a number m divisible by 5^n consisting of n odd digits.

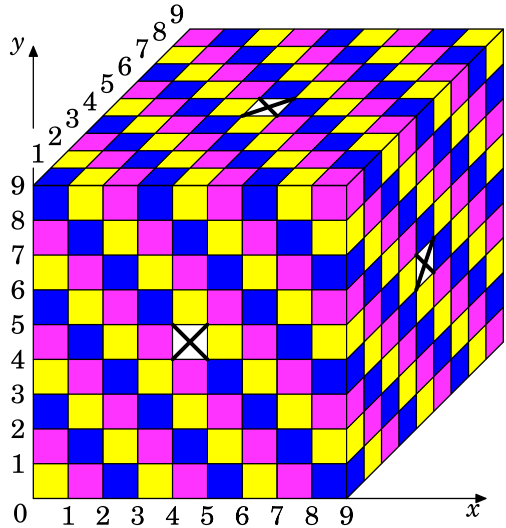
Let us take $q = m/5^n$ and choose $d \in \{1, 3, 5, 7, 9\}$ such that such that $5 \mid d \cdot 2^n + q$. So the number

$$m' = 5^n(d \cdot 2^n + q) = d \cdot 10^n + m$$

is divisible by 5^{n+1} and consists of all n odd digits of m and an odd digit d.

A–T–2

Let us place our cube as shown at the picture. Each of the small cubes can be uniquely determined by three coordinates $[x, y, z]$ in a similar manner as points. The cube with vertex at $(0, 0, 0)$ gets coordinates $[0, 0, 0]$ and the cube with vertex $(9, 9, 9)$ gets coordinates $[8, 8, 8]$. Let us color each of small cubes with one of three colors depending of the remainder obtained in division of sum of its coordinates $x + y + z$ by 3.



A cube F consists of the same number of small cubes of each color.

The cubes removed from F have coordinates $[4,4,0]$, $[4,4,8]$, $[0,4,4]$, $[8,4,4]$, $[4,0,4]$, $[4,8,4]$ and the remainders obtained dividing these sums by 3 are as follows: 2, 1, 2, 1, 2, 1. So in the solid G there are more small cubes for which the corresponding remainder is 0 than these with remainder 1 and 2.

On the other hand each of the rectangular cuboids can only have three small cubes of three distinct colors. Thus it is not possible to build G from such cuboids.

A–T–3

It is easy to see that the polynomial P has degree equal to at least 2.

Let k, n be positive integers, $n \geq 2$, $n > k \geq 0$ and polynomial P is of the form

$$P(x) = a_n x^n + a_k x^k + a_{k-1} x^{k-1} + a_{k-2} x^{k-2} + \ldots + a_1 x + a_0$$

where a_i are real coefficients ($a_n \neq 0$). Then

$$P(x^2) = a_n x^{2n} + a_k x^{2k} + a_{k-1} x^{2(k-1)} + a_{k-2} x^{2(k-2)} + \ldots + a_1 x^2 + a_0,$$

$$P(2x) = 2^n a_n x^n + 2^k a_k x^k + 2^{k-1} a_{k-1} x^{k-1} + 2^{k-2} a_{k-2} x^{k-2} + \ldots + 2a_1 x + a_0.$$

The highest power of x on the left side of the equation is x^{2n}, the second highest power is $x^{\max\{2k,3\}}$. The highest power of x on the right side is x^{2n}, the second highest is x^{n+k}. Since $n+k > 2k$ we can see that for $a_k \neq 0$, $n+k = 3$, and therefore $n = 2$, $k = 1$ or $n = 3$, $k = 0$.

Comparing coefficients at x^{2n} on the both sides we further obtain $a_n = 2^n a_n^2$ and from $a_n \neq 0$ it follows that $a_n = 1/2^n$ must hold.

In the case $n = 3$ and $k = 0$ (and $a_k = 0$) the desired polynomial $P(x)$ is in the form $P(x) = \frac{1}{8}x^3 + A$, where A is a real number. Substituting in the original equation we have

$$\tfrac{1}{8}x^6 + A - 3x^3 + 15x^2 - 24x + 12 = (\tfrac{1}{8}x^3 + A)(x^3 + A) = \tfrac{1}{8}x^6 + \tfrac{9}{8}Ax^3 + A^2.$$

Comparing coefficients at x^2 we obtain $15 = 0$, so this case is impossible.

In the case $n = 2$ nad $k = 1$ the desired polynomial $P(x)$ is in the form $P(x) = \frac{1}{4}x^2 + Ax + B$, where A and B are real numbers. Substituting to the original equation we have

$$\begin{aligned}\tfrac{1}{4}x^4 + Ax^2 + B - 3x^3 + 15x^2 - 24x + 12 &= (\tfrac{1}{4}x^2 + Ax + B)(x^2 + 2Ax + B)\\ &= \tfrac{1}{4}x^4 + \tfrac{3}{2}Ax^3 + (\tfrac{5}{4}B + 2A^2)x^2 + 3ABx + B^2.\end{aligned}$$

Comparing coefficients for all x^i we obtain

$$\tfrac{1}{4} = \tfrac{1}{4}, \quad -3 = \tfrac{3}{2}A, \quad A + 15 = \tfrac{5}{4}B + 2A^2, \quad -24 = 3AB, \quad B + 12 = B^2.$$

Solving this system yields $A = -2$, $B = 4$.

Conclusion. There exists a unique polynomial P satisfying the original equation, namely $P(x) = \frac{1}{4}x^2 - 2x + 4$.

Category B (Individual Competition)

B–I–1

After easy manipulation we can rewrite the given inequality equivalently in the equivalent form

$$(a-2b)^2+(b-2c)^2 \geq 0.$$

Thus, the given inequality holds for all real numbers a, b, c.

Equality holds if and only if $a = 2b$ and simultaneously $b = 2c$, i.e. for the triples $(a,b,c) = (4c,2c,c)$ with arbitrary real number c.

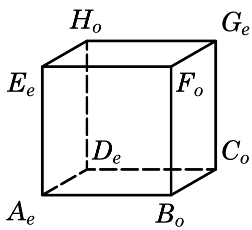
B–I–2

We show, that vertices of each face of the cube $ABCDEFGH$ are necessary assigned exactly two even and two odd numbers from the given set of numbers with numbers of the same parity always in opposite vertices in each face.

It is easy to see, that to vertices of each face it cannot be assigned either at most one even number or least three even numbers. In these cases, for the assignment of vertices in the opposite face of the cube it should remain in that case at most one odd or at most one even number, which is impossible – with respect to conditions of the given problem. Therefore, vertices of each face of this cube must be assigned exactly by two even and exactly two odd numbers.

If we assign an even number to the vertex A, writing A_e, and similarly an odd number to the vertex B, writing B_o.

We can first suppose, that the vertex A is assigned an even number. Then the seven of the vertices of the cube must be assigned even or odd numbers from the given set only in following way (see picture).



The number of assignments of numbers of the set $\{1, 2, \ldots, 8\}$ to the vertices of the cube is then equal (using the combinatorial product principle) to

$$4! \cdot 4! = (4!)^2 = 576.$$

We can similarly proceed, the vertex A is assigned an odd number. We then obtain the same number of corresponding positions.

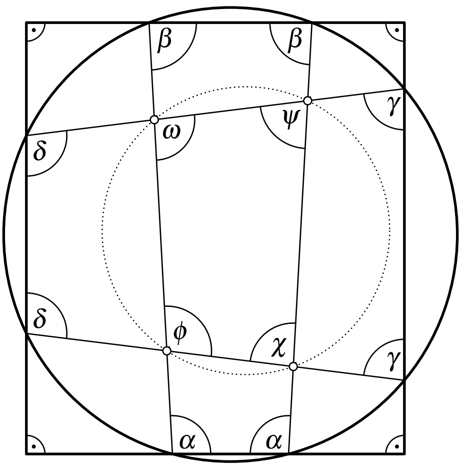
Conclusion. There exist $2 \cdot (4!)^2 = 1\,152$ assignments to vertices of the cube $ABCDEFGH$ by numbers from the set $\{1, 2, \ldots, 8\}$ satisfying conditions of the given problem.

B–I–3

Both trapezoids must be equilateral, so let us denote the angles of one trapezoid by α and β ($\alpha + \beta = 180°$) and of the second one by γ, δ ($\gamma + \delta = 180°$) as shown in the picture.

One can easily compute all the angles of a quadrilateral being the intersection of these two trapezoids

$$\begin{aligned}
&\phi = 360° - 90° - (180° - \alpha) - (180° - \delta) = \alpha + \delta - 90°,\\
&\chi = \alpha + \gamma - 90°,\\
&\psi = \beta + \gamma - 90°,\\
&\omega = \beta + \delta - 90°.
\end{aligned}$$



So, we have

$$\phi + \psi = (\alpha + \delta - 90^\circ) + (\beta + \gamma - 90^\circ) = (\alpha + \beta) + (\gamma + \delta) - 180^\circ = 180^\circ,$$

which proves our claim.

B–I–4

We have $2016 = 2^5 \cdot 3^2 \cdot 7$. The remainder of an integer x^2 after division by 2016 is 1 if and only if 2016 divides $(x-1)(x+1)$. Let D be a common divisor of $x-1$ and $x+1$. Then D divides $(x+1)-(x-1) = 2$, so $D \leq 2$. This means that at most one of the numbers $x-1$ and $x+1$ is divisible by 3 or by 7. If one of the numbers $x-1$ and $x+1$ is divisible by 2^k where $k \geq 2$, the second one is divisible by 2 only in the power 1. So, in order 2^5 to divide $(x-1)(x+1)$ it is necessary and sufficient that 2^4 divides one of the factors.

To $2016 \mid (x-1)(x+1)$ it is necessary and sufficient that some of the 3 numbers $\{2^4, 3^2, 7\}$ divide $(x-1)$ and the others divide $(x+1)$. This yields $2^3 = 8$ combinations altogether. Each of the previous possibilities gives (by the Chinese remainder theorem) a unique solution modulo $2^4 \cdot 3^2 \cdot 7 = 1008$, and therefore exactly 2 solutions in the set $\{1, 2, \ldots, 2016\}$.

There are $8 \cdot 2 = 16$ numbers of the set $\{1, 2, \ldots, 2016\}$ with remainder 1 after division their square by 2016.

Remark. These numbers are 1 ($1008 \mid 0$), 127 ($3^2 \cdot 7 \mid 126$ and $2^4 \mid 128$), 433 ($2^4 \cdot 3^2 \mid 432$ and $7 \mid 434$), 449 ($2^4 \cdot 7 \mid 448$ and $3^2 \mid 450$), 559 ($3^2 \mid 558$ and $2^4 \cdot 7 \mid 560$), 575 ($7 \mid 574$ and $2^4 \cdot 3^2 \mid 576$), 881 ($2^4 \mid 880$ and $3^2 \cdot 7 \mid 882$), 1007 ($1008 \mid 1008$), 1009, 1135, 1441, 1457, 1567, 1583, 1889 and 2015.

Category B (Team Competition)

B–T–1

A 4-digit palindromic number $\overline{abba}$ can be written as

$$a \cdot 1001 + b \cdot 110 = 11(91a + 10b).$$

If such a number is a perfect square, we must have $11 \mid 91a + 10b$, or

$$91a + 10b \equiv 3a - b \equiv 0 \pmod{11}.$$

There are only 8 pairs of digits satisfying this, and possible numbers are therefore $(1;3)$, $(2;6)$, $(3;9)$, $(4;1)$, $(5;4)$, $(6;7)$, $(8;2)$ and $(9;5)$. Checking the prime decompositions of the resulting numbers, we see that $8228 = 2^2 \cdot 11^2 \cdot 17$, and therefore $17 \cdot 8228 = (2 \cdot 11 \cdot 17)^2 = 374^2$ is such a number. Checking all other possibilities confirms that there are no other numbers with the required property.

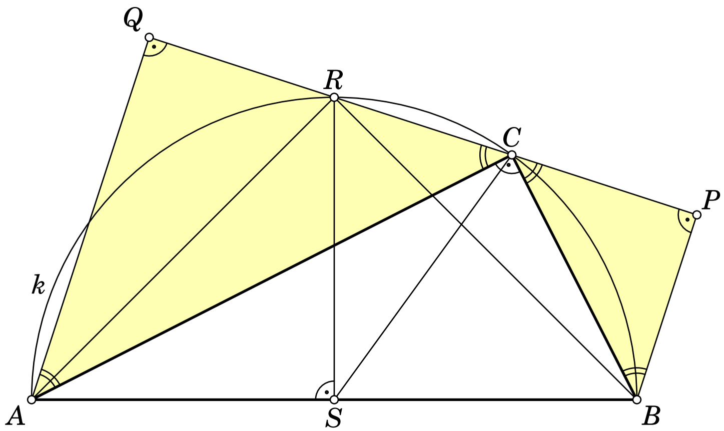
B–T–2

Firstly we can see that the vertex C of the given right-angled triangle is a point of the line PQ, since $|\angle QCA| = |\angle PCB| = 45^\circ$. Therefore

$$|\angle QCP| = |\angle QCA| + |\angle ACB| + |\angle BCP| = 45^\circ + 90^\circ + 45^\circ = 180^\circ.$$

If the given right-angled triangle ABC is isosceles, the statement of the problem is obviously fulfilled.

Without loss of generality we can assume $|AC| > |BC|$ (see picture). Let us consider the intersection point R ($R \neq C$) of the line PQ with the Thales circle k with the center S and the diameter AB. We shall prove that $R = D$.



From the conditions of the problem and from the description of the point R it follows that

$$|\angle RCA| = |\angle QCA| = 45^\circ$$

holds. Moreover, from well-known relation between the angles RCA and RSA in the circle we have

$$|\angle RSA| = 2 \cdot |\angle RCA| = 2 \cdot 45^\circ = 90^\circ.$$

This immediately implies that the point R (on the Thales circle k) also lie on the perpendicular bisector of the hypotenuse AB, i.e. ABR is the right-angled isosceles triangle with hypotenuse AB, and thus $R = D$.

Thus the proof is finished.

B–T–3

It is easy to show that the squares of integers not divisible by 7 can give only remainders 1, 2, 4 when divided by 7. A sum of integers is divisible by 7 if the sum of their remainders when divided by 7 is divisible by 7.

We will call an integer whose square gives a remainder r when divided by 7 *an integer with square-remainder r*. We will further call the sum of squares of integers *a square-sum of these integers*. When the set $\mathcal{A}$ contains six numbers whose square-remainders are 1, then it is not possible to find a subset with square-sum divisible by 7.

We show that if the set $\mathcal{A}$ contains 7 elements or more, then it is possible. First, observe that if the set $\mathcal{A}$ contains three integers with square-remainders of 1, 2 and 4, then we can choose a set consisting of these numbers, because $1 + 2 + 4 = 7$. Similarly, if there are in A seven integers with identical square-remainders, then we can choose as a subset of $\mathcal{A}$ all of these integers.

The following table shows the other cases (in which there are only two of three square-remainders)

Number of square-remainders			Possible choice of square-remainders in a subset
1	2	4	
1	6	0	$1+2+2+2$
2	5	0	$1+2+2+2$
3	4	0	$1+2+2+2$
4	3	0	$1+2+2+2$
5	2	0	$1+1+1+1+1+2$
6	1	0	$1+1+1+1+1+2$
0	1	6	$2+4+4+4$
0	2	5	$2+4+4+4$
0	3	4	$2+4+4+4$
0	4	3	$2+4+4+4$
0	5	2	$2+2+2+2+2+4$
0	6	1	$2+2+2+2+2+4$
1	0	6	$1+4+4+4+4+4$
2	0	5	$1+4+4+4+4+4$
3	0	4	$1+1+1+4$
4	0	3	$1+1+1+4$
5	0	2	$1+1+1+4$
6	0	1	$1+1+1+4$

Category C (Individual Competition)

C–I–1

a) It is easy to see, that for $p = 0$ the given inequality is obviously fulfilled for all real values of a and b.
b) We can easily prove that for $p = 2$ the given inequality is true for all real numbers a and b. We can rewrite the given inequality in the equivalent form $a^2 - 2ab + 2b^2 \geq 0$. Then we have

$$a^2 - 2ab + 2b^2 = (a^2 - 2ab + b^2) + b^2 = (a - b)^2 + b^2 \geq 0,$$

which is obviously true for all real numbers a and b. Thus, the the given inequality is true for $p = 2$ and for all real values of a and b.

If $p = 3$, the given inequality is not satisfied for all real values of a and b. Choosing $a = 3$ and $b = 2$ (for instance) we can see

$$a^2 + 2b^2 = 3^2 + 2 \cdot 2^2 = 17 < 18 = 3 \cdot 3 \cdot 2 = 3ab.$$

C–I–2

Let us denote the three colours by capitals, R (red), W (white) and B (blue). Let us first assume that the vertex A of the considered hexagon $ABCDEF$ is coloured red (**R**). All possibilities fulfilling the conditions of the given problem are listed in the table below (the vertex A is repeated in its right column).

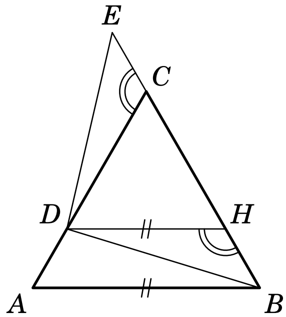
A	B	C	D	E	F	A
R	W	**R**	B	W	B	**R**
R	B	**R**	W	B	W	**R**
R	W	B	**R**	W	B	**R**
R	W	B	**R**	B	W	**R**
R	B	W	**R**	W	B	**R**
R	B	W	**R**	B	W	**R**
R	B	W	B	**R**	W	**R**
R	W	B	W	**R**	B	**R**

We therefore obtain a total of 8 possibilities for the red coloured vertex A. Similarly, if we colour the vertex A white and then blue, we obtain further $8 + 8 = 16$ possibilities colouring all vertices according the conditions of the problem.

Conclusion. There exist $3 \times 8 = 24$ possible colourings of all vertices of hexagon $ABCDEF$ in total.

C–I–3

Let H be the point on the segment BC such that $DH \parallel AB$ (see picture). We prove that the triangles BHD and CED are congruent.



Since $DH \parallel AB$, triangles ABC and CDH are similar. Hence CDH is also an equilateral triangle. Therefore $|DH| = |DC|$. Further, we obviously have

$$|\angle BHD| = |\angle DCE| = 120^\circ \qquad \text{and} \qquad |HB| = |DA| = |EC|.$$

This implies that the triangles BHD and CED are congruent by the theorem *s–a–s* and therefore $|DB| = |ED|$.

This concludes the proof.

C–I–4

Since

$$4 \cdot \frac{x+y}{2} = 5 \cdot \sqrt{xy}$$

is given, we have $2x + 2y = 5\sqrt{xy}$ or $4x^2 - 17xy + 4y^2 = 0$. This is equivalent to $(4x - y)(x - 4y) = 0$, and we see that $x : y = 4 : 1$ or $x : y = 1 : 4$.

Category C (Team Competition)

C–T–1

For arbitrary real numbers a, b, c with $abc \neq 0$, we compute

$$
\begin{aligned}
P = A \cdot B \cdot C &= \frac{a^2+b^2}{c^2} \cdot \frac{b^2+c^2}{a^2} \cdot \frac{c^2+a^2}{b^2} = \\
&= \left(\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)\left(\frac{b^2}{a^2}+\frac{c^2}{a^2}\right)\left(\frac{c^2}{b^2}+\frac{a^2}{b^2}\right) = \\
&= \frac{b^2c^2}{c^2b^2}+\frac{b^2a^2}{c^2b^2}+\frac{c^2}{b^2}+\frac{a^2}{b^2}+\frac{b^4c^2}{a^2c^2b^2}+\frac{b^4a^2}{a^2c^2b^2}+\frac{b^2c^2}{a^2b^2}+\frac{b^2a^2}{a^2b^2} = \\
&= 1+\frac{a^2}{c^2}+\frac{c^2}{b^2}+\frac{a^2}{b^2}+\frac{b^2}{a^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+1 = \\
&= 2+\left(\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)+\left(\frac{b^2}{a^2}+\frac{c^2}{a^2}\right)+\left(\frac{c^2}{b^2}+\frac{a^2}{b^2}\right) = \\
&= 2+A+B+C = 2+S.
\end{aligned}
$$

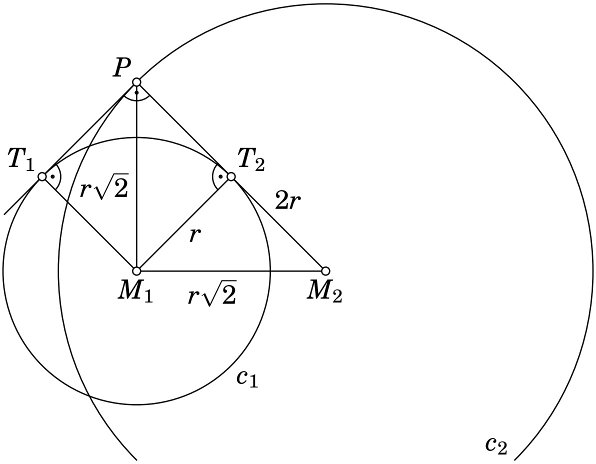
This implies $P-S=2$.

Conclusion. The difference $P-S=2$ holds for all real numbers a, b, c with $abc \neq 0$.

C–T–2

Let T_1 and T_2 be the points of tangency of the tangents of c_1 from P. Since $|\angle M_1T_1P| = |\angle M_1T_2P| = |\angle T_1PT_2| = 90^\circ$ and $|M_1T_1| = |M_1T_2| = r$, we see that $M_1T_1PT_2$ is a square with sides of length r. Since M_1P is a diagonal of this square, we have $|M_1P| = r\sqrt{2}$. We are given that $|M_1M_2| = r\sqrt{2}$ and since P lies on c_2, we have $|M_2P| = 2r$. Since $|M_1P|^2 + |M_1M_2|^2 = 2r^2 + 2r^2 = 4r^2 = |M_2P|^2$, triangle M_1M_2P is right-angled with $|\angle M_2M_1P| = 90^\circ$, and the area $\mathcal{A}$ of M_1M_2P is therefore equal to

$$\mathcal{A} = \frac{1}{2} \cdot |M_1P| \cdot |M_1M_2| = \frac{1}{2} \cdot r\sqrt{2} \cdot r\sqrt{2} = r^2.$$



C–T–3

We note that 5040 has the prime decomposition $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$.

a) $5040 = 7 \cdot 8 \cdot 9 \cdot 10$.
b) $5040 = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$.
c) Since $5040 = 70 \cdot 72$, it is the product of two consecutive even integers. If it were also the product of two consecutive integers, one must be larger than 70 and the other smaller than 72. These cannot be consecutive, and the proof is complete.

Results

Category A (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Marian Poljak	Přerov	8	8	8	6	**30**
2.	Jan Gocník	Přerov	8	8	8	0	**24**
3.	Konstantin Andritsch	Graz	7	1	6	8	**22**
4.	Adam Szreter	Chorzów	8	1	0	6	**15**
	Damian Wałoszek	Bílovec	1	8	3	3	**15**
6.	Jan Równicki	Chorzów	7	0	1	6	**14**
7.	Benedikt Andritsch	Graz	7	1	1	3	**12**
8.	Adrian Steinmann	Graz	8	0	1	1	**10**
9.	Daniel Horiatakis	Graz	8	0	0	1	**9**
	Jiří Nábělek	Bílovec	6	0	2	1	**9**
11.	Jan Šuta	Přerov	8	0	0	0	**8**
12.	Bára Tížková	Bílovec	3	1	1	1	**6**
13.	Martin Kubeša	Přerov	0	0	1	0	**1**
	Tomáš Nguyen	Bílovec	0	0	1	0	**1**
	Orest Tereszczuk	Chorzów	1	0	0	0	**1**
16.	Karol Pieczka	Chorzów	0	0	0	0	**0**

Category B (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Denisa Chytilová	Přerov	8	8	8	1	**25**
2.	Marcin Socha	Chorzów	8	8	8	0	**24**
3.	Katarzyna Kępińska	Chorzów	8	8	6	1	**23**
4.	Sebastian Fellner	Graz	8	8	0	6	**22**
5.	Jana Pallová	Přerov	8	6	1	2	**17**
6.	Olaf Placha	Chorzów	2	6	8	0	**16**
7.	Michaela Svatošová	Bílovec	3	8	1	1	**13**
8.	Tomáš Křižák	Bílovec	2	8	0	0	**10**
9.	Oskar Kleinoscheg	Graz	0	8	0	1	**9**
10.	Thiemo Dsubanko	Graz	0	8	0	0	**8**
	Jakub Gogela	Přerov	0	8	0	0	**8**
12.	Berenika Čermáková	Bílovec	2	2	0	0	**4**
13.	Kristýna Janovská	Bílovec	2	0	0	0	**2**
14.	Julian Narimany	Chorzów	0	1	0	0	**1**
15.	Louies Kirolos	Graz	0	0	0	0	**0**

Category C (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Amálie Dostalíková	Přerov	3	8	0	0	**11**
	Clara Rinner	Graz	3	0	8	0	**11**
3.	Max Dresler	Bílovec	1	2	1	6	**10**
4.	Jaroslav Hradil	Přerov	1	1	0	6	**8**
	Martin Šváb	Přerov	0	1	0	7	**8**
6.	Ondřej Jozek	Bílovec	2	4	0	0	**6**
7.	Matyáš Florík	Bílovec	0	0	0	5	**5**
8.	Minh Nguyen	Graz	2	0	0	2	**4**
9.	Bartosz Kipa	Chorzów	2	0	0	1	**3**
10.	Darian Poljak	Přerov	2	0	0	0	**2**
	Johannes Trattner	Graz	2	0	0	0	**2**
12.	Štěpán Postava	Bílovec	0	0	0	1	**1**
13.	Flavio Arrigoni	Graz	0	0	0	0	**0**
	Pola Gajek	Chorzów	0	0	0	0	**0**
	Wojciech Socha	Chorzów	0	0	0	0	**0**
	Anna Szymik	Chorzów	0	0	0	0	**0**

Category A (Team Competition)

Rank	School	1	2	3	∑
1.	GJŠ Přerov	8	8	1	**17**
2.	AZSO Chorzów	0	1	3	**4**
3.	GMK Bílovec	0	0	3	**3**
4.	BRG Kepler, Graz	0	0	1	**1**

Category B (Team Competition)

Rank	School	1	2	3	∑
1.	GJŠ Přerov	8	8	6	**22**
2.	AZSO Chorzów	8	8	4	**20**
3.	GMK Bílovec	8	7	2	**17**
4.	BRG Kepler, Graz	8	8	0	**16**

Category C (Team Competition)

Rank	School	1	2	3	∑
1.-2.	BRG Kepler, Graz	8	4	8	**20**
1.-2.	GMK Bílovec	8	4	8	**20**
3.	AZSO Chorzów	1	8	8	**17**
4.	GJŠ Přerov	0	4	6	**10**

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