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MATHEMATICAL DUEL ’17

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Preface

The 25th Mathematical Duel was held in Graz from 7th till 11th March 2017. It was the last of three competitions planned as part of the “Mathematical Duel Plus” project, which was financed by the Erasmus Plus programme. The students from three age groups – Category A (age 17–19), Category B (age 15–17), Category C (age 13–15) – took part in individual and team competition. The teams came from Czech, Polish and Austrian schools, namely Gymnázium Jakuba Škody in Přerov and Gymnázium Mikuláše Koperníka in Bílovec, Akademicki Zespół Szkół Ogólnokształcących in Chorzów, Bundesrealgymnasium Kepler in Graz.

The individual competition started on March 8th at 9 a.m. The students in each age group were allowed two hours and thirty minutes to solve four problems (they could achieved maximum score of 8 points for each problem). Since all problems were stated in English, the teachers aided the members of their respective teams in understanding their content. The students were allowed to write their solutions either in their native languages or in English.

The team competition started after a coffee and snack break. The participating schools were allowed to send 12 persons—four persons for each age group. Thus, four students representing each school and age group formed one team. The teams had 100 minutes to solve three problems.

This booklet contains all problems with solutions and results of the 25th Mathematical Duel from the year 2017.

The authors

Problems

Category A (Individual Competition)

A–I–1

Determine all integers n satisfying the equation

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}\,\sqrt{k+1}\,(\sqrt{k}+\sqrt{k+1})} = \frac{24}{25}\,.$$

Pavel Calábek

A–I–2

Prove that there exist infinitely many natural powers of 4 such that their decimal representations contain at least one odd digit.

Jacek Uryga

A–I–3

Let a, b, c be lengths of the sides of a triangle ABC and h_a, h_b, h_c be the lengths of its altitudes from the vertices A, B, C, respectively. Furthermore, let d be the diameter of the circumcircle of the triangle ABC. Prove that the inequality

$$\frac{a^2+b^2+c^2}{h_a+h_b+h_c} \geq d$$

holds. When does equality hold?

Jaroslav Švrček

A–I–4

Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ which take finitely many values and satisfy the condition

$$f(x+1) = 2f(x) + 1$$

for all real numbers x.

Jacek Uryga

Category A (Team Competition)

A–T–1

Determine all quadruples (x, y, z, u) of non-negative integers satisfying the equation
$$2^x + 3^y + 5^z = 7^u.$$

Józef Kalinowski

A–T–2

We are given two mutually perpendicular rays p, q in the plane with common initial point A. Let C be an interior point of the smaller angle with the arms p and q. Determine the locus of the centres of the circumcircles of all cyclic quadrilaterals $ABCD$ with $B \in p$ and $D \in q$.

Jaroslav Švrček

A–T–3

Determine all integers n such that $4n^2 + 5n + 16$ is a perfect square.

Pavel Calábek

Category B (Individual Competition)

B–I–1

Let n be a positive integer and $a_1, a_2, \ldots, a_n$ be real numbers fulfilling the condition

$$a_1 \geq a_2 \geq \ldots \geq a_n.$$

Let us define $S_i = a_1 + a_2 + \ldots + a_i$ for all i $(2 \leq i \leq n)$. Prove that the inequality

$$\frac{S_{n-1}}{n-1} \geq \frac{S_n}{n}$$

holds. When does equality hold?

Józef Kalinowski

B–I–2

We are given a triangle ABC with the longest side AB. Let K, L be points on the side AB such that $|AL| = |AC|$, $|BK| = |BC|$, and M, N be points on the sides BC, AC, respectively, such that $|BM| = |BL|$, $|AN| = |AK|$. Prove that the points K, L, M, N and C lie on the same circle.

Jaroslav Švrček

B–I–3

Prove that there exist infinitely many natural powers of 9 such that their decimal representations contain at least one even digit.

Jacek Uryga

B–I–4

On the circle with radius 1 we are given a set of arcs with lengths less than π, the sum of whose is greater than 2π. Prove that there exists a straight line that goes through the center of the circle and intersects at least three of these arcs.

Jacek Uryga

Category B (Team Competition)

B–T–1

We are given two sets $\mathsf{M} = \{25^{25}, 25^{25}+1, 25^{25}+2, 25^{25}+3\}$ and $\mathsf{N} = \{a, b, c, d\}$. Determine the number of all bijective (i.e. unique and invertable) mappings $\mathsf{M} \to \mathsf{N}$ such that the sum

$$ab + bc + cd + da$$

assumes the maximum possible value and determine this value.

Jaroslav Švrček

B–T–2

In an equilateral triangle ABC, let M denote the midpoint of the side AB. Furthermore, let F denote the foot of the perpendicular from M to BC. The line ℓ is perpendicular to AF and passes through C. Prove that ℓ intersects MF in the midpoint of MF.

Robert Geretschläger

B–T–3

Albert and Zita are playing a game. On the table in front of them, there is a bowl with n coins. In each move, they are allowed to take 1, 2 or 3 coins from the bowl. They are never allowed to take the same number that the other has taken in the move before. (If Zita takes 2 coins, for example, Albert can only take either 1 or 3 coins in the next move.) Albert has the first move, and after that, they take turns. If Zita ever has a total number of coins divisible by 3, she wins. The game ends if Zita wins or when there are no more moves possible for the player whose turn would be next. Albert wins if Zita does not. Which player will win the game if both play with the best possible strategy? Describe their strategy and prove that this player will certainly win. For which values of n does Albert have a winning strategy, and for which values of n does Zita have a winning strategy?

Robert Gererschläger

Category C (Individual Competition)

C–I–1

How many integers from the set $\{1, 2, \ldots, 2017\}$ are divisible by 11 but not by 7?

Józef Kalinowski

C–I–2

We are given a rectangle $ABCD$ with $|AB| = a > b = |BC|$. Let us suppose that the feet of the perpendiculars from the vertices A and C to the diagonal BD divide this diagonal into three congruent segments. Determine the ratio $a : b$.

Jaroslav Švrček

C–I–3

We are given a system of equations

$$\begin{aligned} x + cy &= c, \\ 2x + 4y &= 3 \end{aligned}$$

with unknowns x and y. Determine all possible values of the real parameter c such that the equation $4x - y = 2$ holds.

Józef Kalinowski

C–I–4

The Count of Lichenem likes to count, but he doesn't like most of the numbers. He likes a number if it has both even and odd digits, and he doesn't like a number if it has an even number of odd digits or an odd number of even digits.

a) How many numbers smaller than 100 does the Count like?
b) How many numbers smaller than 1 000 does the Count like?
c) How many numbers smaller than 10 000 does the Count like?

Robert Geretschläger

Category C (Team Competition)

C–T–1

Prove that for any odd integer n, the integer

$$n^3 + 3n^2 - n - 3$$

is divisible by 24.

Józef Kalinowski

C–T–2

Let $ABCD$ be a rectangle and M a point on the side CD such that ABM is a right-angled triangle with the hypotenuse AB. It is known that $|AB| : |BC| = 3 : \sqrt{2}$ holds and $|CM| > |DM|$. Let P, Q and R denote the areas of triangles AMD, BMC and ABM, respectively. Determine the ratio $P : Q : R$.

Robert Geretschläger

C–T–3

Different positive integers written on the blackboard were divided into three non-empty sets. Numbers of the first set were multiplied by 2, numbers of the second set were multiplied by 3, numbers of the third set were multiplied by 5 and all numbers written initially on the blackboard were erased. After this, only four different two-digit numbers remained on the blackboard. How many integers were initially on the blackboard? Determine all possible answers.

Pavel Calábek

Solutions

Category A (Individual Competition)

A–I–1

We can rewrite each of fractions in the given sum in the following manner

$$\frac{1}{\sqrt{k}\,\sqrt{k+1}\,(\sqrt{k}+\sqrt{k+1})} = \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\,\sqrt{k+1}} = \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$$

for all positive integers k. We therefore obtain

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}\,\sqrt{k+1}\,(\sqrt{k}+\sqrt{k+1})} = \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = 1 - \frac{1}{\sqrt{n+1}}.$$

Solving the equation

$$1 - \frac{1}{\sqrt{n+1}} = \frac{24}{25}$$

give us $\sqrt{n+1} = 25$ and therefore $n = 624$.

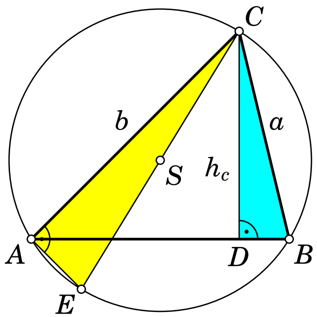
Conclusion. The given equation has a unique solution, namely $n = 624$.

A–I–2

The last digit of a natural power of 4 can be only 4 or 6. By the divisibility rule for 4, the last two digits can only be: 04, 16, 24, 36, 44, 56, 64, 76, 84, 96. The endings 16, 36, 56, 79, 96 contain an odd tens digit. For the powers with endings 04, 24, 44, 64, 84, the next larger power of 4 therefore contains an odd tens digit. We see that every larger power of 4 therefore contains an odd tens digit, and the proof is complete.

A–I–3

We will prove the given inequality for an acute-angled triangle ABC (The proofs for a right-angled or obtuse-angled triangle ABC are similar). Let D denote the foot of the altitude from the vertex C (D is an interior point of the side AB). Furthermore, let CE be the diameter of the circumcircle of the given triangle ABC ($|CE| = d$).



From the picture we see that $|\angle ABC| = |\angle AEC|$. We therefore have $|\angle ACE| = |\angle DCB|$, and the right triangles ECA and BCD are therefore similar. This implies

$$|AC| : |EC| = |DC| : |BC|, \qquad \text{or} \qquad b : d = h_c : a.$$

We can rewrite the last identity in the form

$$ab = dh_c. \tag{1}$$

Similarly, we obtain

$$bc = dh_a \tag{2}$$

and

$$ca = dh_b. \tag{3}$$

Adding identities (1)–(3) we get

$$ab + bc + ca = d(h_a + h_b + h_c).$$

From the well-known inequality $a^2 + b^2 + c^2 \geq ab + bc + ca$ (with equality for $a = b = c$), we therefore obtain

$$a^2 + b^2 + c^2 \geq ab + bc + ca = d(h_a + h_b + h_c),$$

which is equivalent to the desired inequality

$$\frac{a^2 + b^2 + c^2}{h_a + h_b + h_c} \geq d.$$

Equality holds iff $a = b = c$, i.e. in the case of the equilateral triangle ABC. Thus, the proof is finished.

Another solution. Rewriting the given inequality we have

$$a^2 + b^2 + c^2 \geq d(h_a + h_b + h_c) = 2r(h_a + h_b + h_c), \tag{4}$$

where r is the radius of the circumcircle of the triangle ABC. The Law of Sines yields

$$2r = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}. \tag{5}$$

Substituting in (4) from (5) we obtain

$$a^2 + b^2 + c^2 \geq \frac{b}{\sin\beta} \cdot h_a + \frac{c}{\sin\gamma} \cdot h_b + \frac{a}{\sin\alpha} \cdot h_c. \tag{6}$$

Since

$$\frac{h_a}{\sin\beta} = c, \quad \frac{h_b}{\sin\gamma} = a, \quad \frac{h_c}{\sin\alpha} = b,$$

we can rewrite (6) in the form

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

which is an equivalent form, as in the first solution.

A–I–4

Let us take an arbitrary number a and define a sequence (x_n) as follows

$$x_1 = a, \qquad x_{n+1} = x_n + 1.$$

Note that the sequence $a_n = f(x_n)$ satisfies the condition

$$a_{n+1} = f(x_{n+1}) = f(x_n + 1) = 2a_n + 1$$

and we therefore have $a_{n+1} - a_n = a_n + 1$ for every n.

Now, observe that if $a_1 > -1$, it follows that $a_2 - a_1 = a_1 + 1 > 0$ holds. Thus we have $a_2 > -1$ and $a_3 - a_2 = a_2 + 1 > 0$. Repeated application of this argument yields

$$-1 < a_1 < a_2 < a_3 < \ldots$$

This means that the set of values of f cannot be finite.

Analogously, we can see that if $a_1 < -1$, then

$$-1 > a_1 > a_2 > a_3 > \dots$$

follows, and again the set of values of f cannot be finite.

It follows that the only function satisfying our assumption can be the constant function $f(x) = -1$, which indeed satisfies the assumption.

Category A (Team Competition)

A–T–1

The numbers 2^x, 3^y, 5^z and 7^u are certainly all integers. Because the number 7^u is an odd number, then the number 2^x also is an odd number. It is possible for $x = 0$ only. Then we get the equation

$$1 + 3^y + 5^z - 7^u = 0. \tag{1}$$

Let us consider two possible cases:

a) If $y \geq 1$, then we have

$$\begin{aligned} 1 - 7^u &\equiv 0 \pmod 3, \\ 3^y + 5^z &\equiv (-1)^z \pmod 3. \end{aligned}$$

Summing up both these congruences we get

$$1 + 3^y + 5^z - 7^u \equiv (-1)^z \pmod 3$$

which is a contradiction to (1). Thus, the equation (1) has no solution in non-negative integers y, z, u.

b) If $y = 0$, we have the equation

$$2 + 5^z = 7^u. \tag{2}$$

Futher let us consider three possible cases in (2):

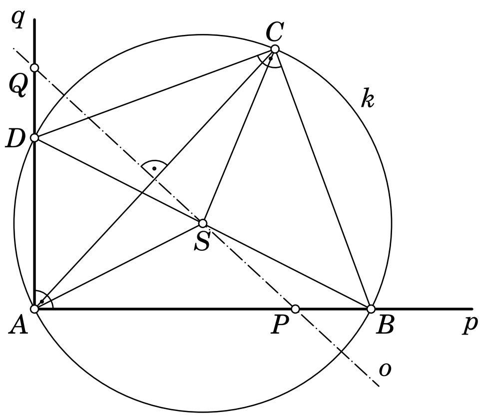
- $z \geq 2$. Then $2 + 5^z \equiv 2 \pmod{25}$ and dividing the number 7^u by 25 we cyclic obtain the remainders: 7, 24, 18 and 1 and the last equation is not fulfilled for any non-negative integer pair of numbers z, u.

- $z = 1$. Then we have the new equation $2 + 5 = 7^u$ and then $u = 1$ and after checking, we have a solution $x = y = 0$, $z = u = 1$ of the equation.
- $z = 0$. Then we have the new equation $2 + 1 = 7^u$, and this equality is not true for any non-negative integer number u.

Conclusion. The given Diophantine equation has the unique solution $x = y = 0, z = u = 1$ (in non-negative integers).

A–T–2

Let us consider a cyclic quadrilateral $ABCD$ with $B \in p$, $D \in q$ and with the circumcenter S. We then have $|\angle BAD| = |\angle BCD| = 90°$ and $|SA| = |SB| = |SC| = |SD|$. The point S must therefore be the midpoint of the segment BD and simultaneously it must lie on the perpendicular bisector o of the segment AC. Let P and Q be the points of intersection of o with the rays p and q, respectively (see picture). This implies that the center S of the circumcircle of the quadrilateral $ABCD$ lies on the segment PQ.



Conversely, we can show that for each interior point S of the segment PQ, there exists a cyclic quadrilateral $ABCD$ with the given properties. Let us consider the circle c with the center S and the radius $|SA| = |SC|$. The circle c meets the rays p, q in points B and D

respectively. Since $|\angle BAD| = 90^\circ$, S is the midpoint of BD. Thus, for each interior point S of the segment PQ, there exists a cyclic quadrilateral $ABCD$ with $B \in p$ and $D \in q$.

Conclusion. The locus of centers of all quadrilaterals $ABCD$ with the given properties is the open segment PQ.

A–T–3

Let

$$4n^2 + 5n + 16 = m^2$$

holds for non-negative integer m. This is equivalent to

$$\left(2n + \frac{5}{4}\right)^2 - m^2 = \frac{25}{16} - 16$$

or

$$(8n + 5 - 4m)(8n + 5 + 4m) = -231.$$

We note that $231 = 3 \cdot 7 \cdot 11$ holds and both factors $8n + 5 - 4m \leq 8n + 5 + 4m$ are congruent to 1 modulo 4. We are therefore left with the following cases.

$8n+5-4m$	-231	-11	-7	-3
$8n+5+4m$	1	21	33	77
$16n+10$	-230	10	26	74
n	-15	0	1	4
m	29	4	5	10

Conclusion. The value of $4n^2 + 5n + 16$ is a prefect square only for $n \in \{-15, 0, 1, 4\}$.

Category B (Individual Competition)

B–I–1

By the assumptions, we have the inequalities

$$\begin{aligned} a_1 &\geq a_n, \\ a_2 &\geq a_n, \\ &\vdots \\ a_{n-1} &\geq a_n. \end{aligned}$$

Summing up, we therefore obtain

$$S_{n-1} \geq (n-1) \cdot a_n.$$

Adding $(n-1) \cdot S_{n-1}$ to both sides of these inequality, we obtain

$$n \cdot S_{n-1} \geq (n-1) \cdot S_n$$

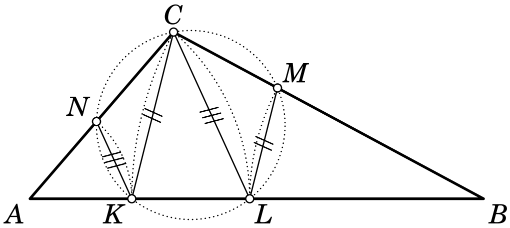
and dividing this inequality by $n(n-1) > 0$ yields the required inequality and finishing the proof.

Equality holds iff $S_{n-1} = (n-1) \cdot a_n$, and this is only possible if equality holds in each of the inequalities $a_1 \geq a_n$, $a_2 \geq a_n, \ldots,$ $a_1 \geq a_n$, which holds iff

$$a_1 = a_2 = \ldots = a_n.$$

B–I–2

It is easy to see that $KLCN$ and $KLMC$ are isosceles trapezoids with $LC \parallel NK$ and $LM \parallel CK$ since triangles ACL, BCK, ANK and ABL are all isosceles. Both of these trapezoids are therefore cyclic. Their vertices lie on the circumcircle of the triangle KLC, because the vertices K, L and C lie simultaneously on the circumcircles of both isosceles trapezoids. Thus, all five points K, L, M, N and C lie on the same circle, and the proof is finished.



B–I–3

The number 9^2 is equal to 81. Its tens digit is even. Multiplying this number by 9 we obtain a number with the last digit 9 and with an even tens digit, namely 2.

If we multiply such a number by 9, we obtain a number with the ones digit 1 and the tens digit even. If we multiply this number by 9 once again, we again obtain a number with the ones digit 9 and the tens digit even. This shows that the tens digit of all powers of 9 are even, completing the proof.

B–I–4

Let A denote the set of n arcs and consider the set A' of all arcs from A reflected with respect to the center of the circle. It is obvious that the sum of the lengths of arcs from the set $A \cup A'$ is greater than 4π and none of these arcs will be intersected more than onece by any straight line going through the center of the circle.

If the assumption were false, each straight line going through the center of the circle would intersect zero or two arcs from $A \cup A'$ on each side of the center of the circle. This would mean in turn that the sum of all arcs from $A \cup A'$ would be less than or equal to 4π, which is a contradiction. We therefore see that there must exist a line with the required property.

Category B (Team Competition)

B–T–1

First, let us write the given sum in the way

$$ab + bc + cd + da = (a + c)(b + d).$$

From the AM-GM inequality for the positive numbers $a+c$ and $b+d$ we therefore get

$$\begin{aligned}(a + c)(b + d) \le \frac{[(a + c) + (b + d)]^2}{4} &= \frac{(a + b + c + d)^2}{4}\\ &= \frac{(4 \cdot 25^{25} + 6)^2}{4}.\end{aligned}$$

This maximum value can be obtained if and only if

$$a + c = b + d. \tag{1}$$

For the given values of M, this is fulfilled only in two cases:

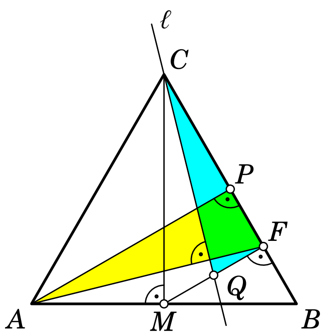
a) $a, c \in \{25^{25}, 25^{25} + 3\}$ and $b, d \in \{25^{25} + 1, 25^{25} + 2\}$,
b) $a, c \in \{25^{25} + 1, 25^{25} + 2\}$ and $b, d \in \{25^{25}, 25^{25} + 3\}$.

In each case we have exactly 4 bijective mappings of both sets fulfilling (1).

Conclusion. There exist altogether $2 \cdot 4 = 8$ bijective mappings satisfying the conditions of the given problem.

B–T–2

Let P denote the midpoint of BC and Q the common point of ℓ and MF.



We first note that triangles APB and CFM are both right triangles with $|\angle BAP| = |\angle MCF| = 30^\circ$, and are therefore similar. We now further note that $AP \perp CB = CF$, $CB \perp MF$ (and thus $PF \perp FQ$), and $CQ = \ell \perp AF$ hold. The triangles APF and CFQ must therefore also be similar. Since $|PF| = \frac{1}{2} \cdot |PB|$, we therefore also have $|FQ| = \frac{1}{2} \cdot |FM|$, and we see that ℓ does indeed intersect MF in the mid-point of MF as claimed.

B–T–3

Albert certainly wins for $n \leq 3$, since Zita will never have 3 coins. If there are 4 coins, Albert can take at least 2 of them in his first move, and Zita again cannot have 3 coins. If there are 5 or 6, Albert can take 3 coins in his first move. Since Zita cannot take 3, Albert can take another, and once again Zita can never get 3 coins altogether. We see that Albert certainly has a winning strategy for $n \leq 6$.

We now assume that n is sufficiently large. For his first move, Albert must take 3, because Zita would otherwise take 3 on her first move and win. If Zita then takes 1, Albert must take 2 (because Zita would otherwise take 2 and have a total of $1 + 2 = 3$ and win). Zita can then take 1 again. Since Albert must now take 2 or 3, he cannot block Zita from taking 1 again on her third move, and she will certainly win. This scenario is true for all values of $n \geq 11$, and Zita will certainly win in this case.

For $7 \leq n \leq 10$, Albert must once again take 3 in the first move. No matter what Zita does, Albert can block her on the next moves, as we see in the following argument.

Zita can only take 1 or 2 coins in her first move, and Albert can block a win in Zita's second move by taking 2 if Zita took 1 and 1 if Zita took 2. No matter how many coins Zita takes in her second move, Albert can take all remaining coins with his third move and win.

We see that Albert has a winning strategy for $n \leq 10$ and Zita has a winning strategy for $n \geq 11$.

Category C (Individual Competition)

C–I–1

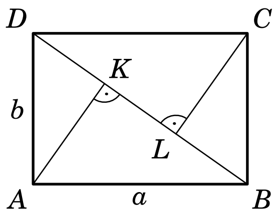
Let $A = \{1, 2, \ldots, 2017\}$. Because $2017 : 11 \doteq 183.3$, there are 183 numbers in the set A what are divisible by 11. Among these are the numbers not fulfilling the assumptions of the problem, namely these divisible divisible by $7 \cdot 11 = 77$. Because $2017 : 77 \doteq 26.2$, there are 26 numbers in A that are divisible by 77.

We therefore have $183 - 26 = 157$ numbers in A that are divisible by 11, but not by 7.

C–I–2

Let K and L be the feet of the perpendiculars from A and C to the diagonal BD, respectively, with

$$|DK| = |KL| = |LB| = \frac{1}{3}\,|BD| = \frac{1}{3}\,\sqrt{a^2 + b^2}.$$



Using the Pythagorean theorem in the right triangles ABK and DAK yields

$$|AK|^2 = a^2 - \frac{4}{9}\,(a^2 + b^2) = b^2 - \frac{1}{9}\,(a^2 + b^2).$$

This implies

$$a^2 : b^2 = 2 : 1, \qquad \text{or} \qquad a : b = \sqrt{2} : 1.$$

Conclusion. The ratio of the lengths of the sides of the considered rectangle is $a : b = \sqrt{2} : 1$.

C–I–3

By the first equation we have $x = c - cy$. Substituting in the second equation, we obtain

$$2(c - cy) + 4y = 3 \Leftrightarrow 2c - 2cy + 4y = 3 \Leftrightarrow y(4 - 2c) = 3 - 2c.$$

If $4 - 2c \neq 0$, we therefore obtain

$$y = \frac{3 - 2c}{4 - 2c}.$$

(Note that $y - 2c = 0 \Leftrightarrow c = 2$ yields $x + 2y = 2$ which contradicts $2x + 4y = 3$.) Then

$$x = c - cy = c(1 - y) = c\left(1 - \frac{3 - 2c}{4 - 2c}\right) = \frac{c}{4 - 2c}.$$

The solution of the system is therefore

$$x = \frac{c}{4 - 2c}, \qquad y = \frac{3 - 2c}{4 - 2c}$$

for $c \neq 2$.

Since x and y satisfy the equation $4x - y = 2$, we have

$$4 \cdot \frac{c}{4 - 2c} - \frac{3 - 2c}{4 - 2c} = 2 \quad \Leftrightarrow \quad \frac{6c - 3}{4 - 2c} = 2.$$

We therefore have $c = \frac{11}{10}$.

Conclusion. The system has a unique solution (x, y) satisfying the equation $4x - y = 2$ for $c = \frac{11}{10}$.

C–I–4

a) The Count does not like any one-digit numbers, since they cannot have both odd and even digits. He does not like any two-digit numbers either, since the only way they can have both odd and even digits is if they have one of each, and this means that they have an odd number (one) of even digits. We see that the Count does not like any numbers smaller than 100.

b) From a), we know that the only numbers smaller than 1000 the Count can like must have three digits. If a three-digit number has both odd and even digits, it must have 2 of one kind and one of the other. Since the count does not like numbers with an odd number of even digits, the ones he likes all have an even number of even digits. These all therefore have 2 even digits and one odd digit. If the odd digit is the first one, it can be either $1, 3, 5, 7$ or 9. The other two digits can each be either $0, 2, 4, 6$ or 8. There are therefore $5^3 = 125$ such numbers. Similarly, if the odd digit is either the second or third digit, it can also be any of the 5 odd digits. The first digit cannot be 0, so it can only be one of the four digits $2, 4, 6$ or 8, while the remaining one can be any of the 5 even digits in either case. In both cases, there are $5 \cdot 4 \cdot 5 = 100$ numbers the Count likes. This yields a total of $125+2\cdot 100 = 325$ numbers less than 1000 the Count likes.

c) We shall now show that the Count does not like any four-digit numbers. If a four-digit number has both odd and even digits, it has either 2 of each, or one of one kind and 3 of the other. If it has two of each, it has an even number of odd digits, and Count does not like it. If it has one of one kind and three of the other, it has either 1 or 3 even digits, i.e. an odd number of even digits in either case. The Count does not like any such a number either. We see that the Count does not like any four-digits numbers, and any number smaller than 10000 he likes, must be smaller than 1000. The number of such numbers was already determined in b) to be equal to 325.

Category C (Team Competition)

C–T–1

Let $L = n^3 + 3n^2 - n - 3$. Factorizing, we obtain

$$\begin{aligned} L &= n^3 + 3n^2 - n - 3 = n^2\,(n + 3) - (n + 3) = (n^2 - 1)(n + 3) \\ &= (n - 1)(n + 1)(n + 3). \end{aligned}$$

Because n is an odd integer, we can write n in the form $n = 2k - 1$, where k is an integer. Substitution then yields

$$\begin{aligned} L &= (2k-1-1)(2k-1+1)(2k-1+3) = (2k-2)\cdot 2k \cdot (2k+2) \\ &= 8(k-1)k(k+1), \end{aligned}$$

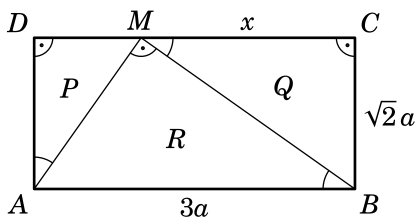
and this means, that L is divisible by 8. It follows that the number $(k-1)k(k+1)$ is the product of three consecutive integers. Among these three consecutive integers, exactly one is divisible by 3. It follows that the product $(k-1)k(k+1)$ is divisible by 3 for any integer k, and the number $8(k-1)k(k+1)$ is therefore divisible by 24, what finishing the proof.

C–T–2

Writing $|AB| = 3a$ and $|BC| = \sqrt{2}\,a$, we can set $|MC| = x$. As $AB \parallel CD$, we have $|\angle CMB| = |\angle ABM|$, and since we also have

$$|\angle DAM| = 90^\circ - |\angle AMD| = 90^\circ - (180^\circ - 90^\circ - |\angle CMB|) = |\angle CMB|,$$

we see that the three right triangles AMD, MBC and BAM are all similar.



We now note that $|MD| = 3a - x$, and therefore

$$\begin{aligned} |BC| : |CM| = |MD| : |DA| \quad &\Leftrightarrow \quad \sqrt{2}\,a : x = (3a - x) : \sqrt{2}\,a \\ &\Leftrightarrow \quad 2a^2 = 3ax - x^2 \end{aligned}$$

which gives us

$$x^2 - 3ax + 2a^2 = 0 \quad \Leftrightarrow \quad (x-a)(x-2a) = 0,$$

and therefore either $x = a$ or $x = 2a$. Since $|CM| > |DM|$ we have $|MC| = x = 2a$, and therefore $|MD| = 3a - 2a = a$. The areas of the triangles can therefore be calculated as

$$P = \tfrac{1}{2} \cdot \sqrt{2}\, a \cdot a = \tfrac{1}{2}\sqrt{2}\, a^2,$$
$$Q = \tfrac{1}{2} \cdot \sqrt{2}\, a \cdot 2a = \sqrt{2}\, a^2$$

and

$$R = |AB| \cdot |BC| - (P + Q) = 3\sqrt{2}\, a^2 - (\tfrac{1}{2}\sqrt{2}\, a^2 + \sqrt{2}\, a^2) = \tfrac{3}{2}\sqrt{2}\, a^2,$$

which gives us

$$P : Q : R = 1 : 2 : 3.$$

C–T–3

Let M_2, M_3 and M_5 denote the sets of numbers initially written on the blackboard which are than multiplied by 2, 3 and 5, respectively. Furthermore, let F denote the final set of four two-digit integers on the blackboard.

Different numbers from each of the sets M_i map onto different numbers in F and therefore each set M_i has at most four elements. Since all M_i are non-empty, each of them has at least one element. Each of the four integers in F has at least one origin on the table. It follows that the original configuration of numbers on the blackboard consist of at most $3 \cdot 4 = 12$ integers.

If at least three integers from F have origins in each of the sets M_2, M_3 and M_5, they are divisible by 2, 3 and 5, and therefore by $2 \cdot 3 \cdot 5 = 30$. Only three such two-digit integers exists, namely 30, 60, 90, and therefore these numbers must lie in F. This means $\{15, 30, 45\} \subset M_2$, $\{10, 20, 30\} \subset M_3$ and $\{6, 12, 18\} \subset M_5$, which is a contradiction, because 30 are in both sets M_2 and M_3. It follows that there could not be 11 or 12 integers on the blackboard initially, because in the first case two sets M_i contain 4 elements and the third 3 elements, and in the second case all M_i contain four elements. In both cases, three numbers from F have origins in all M_i, which is impossible.

The following example shows that there could have been any number of integers from 4 to 10 on the blackboard initially (where # denotes the number of integers initially written on the blackboard).

#	M_2	M_3	M_5	F
4	$30, 45$	10	3	$15, 30, 60, 90$
5	$15, 30, 45$	10	3	$15, 30, 60, 90$
6	$15, 30, 45$	$5, 10$	3	$15, 30, 60, 90$
7	$15, 30, 45$	$5, 10, 20$	3	$15, 30, 60, 90$
8	$15, 30, 45$	$5, 10, 20$	$3, 6$	$15, 30, 60, 90$
9	$15, 30, 45$	$5, 10, 20$	$3, 6, 12$	$15, 30, 60, 90$
10	$15, 30, 45$	$5, 10, 20$	$3, 6, 12, 18$	$15, 30, 60, 90$

Conclusion. Initially, there could be any number from 4 to 10 different integers written on the blackboard.

Results

Category A (Individual Competition)

Rank	Name	School	1	2	3	4	$\sum$
1.	Sebastian Fellner	BRG Kepler	8	8	0	8	**24**
2.	Adam Szreter	AZSO Chorzów	0	6	8	8	**22**
3.	Jan Šuta	GJŠ Přerov	8	8	0	2	**18**
4.	Jiří Nábělek	GMK Bílovec	0	8	6	3	**17**
	Karol Pieczka	AZSO Chorzów	0	8	8	1	**17**
6.	Piotr Breguła	AZSO Chorzów	8	8	0	0	**16**
	Vít Horčička	GJŠ Přerov	8	8	0	0	**16**
	Marcin Socha	AZSO Chorzów	7	8	0	1	**16**
	Bára Tížková	GMK Bílovec	8	7	1	0	**16**
10.	Jana Pallová	GJŠ Přerov	0	8	0	6	**14**
	Damian Wałoszek	GMK Bílovec	0	8	0	6	**14**
12.	Konstantin Andritsch	BRG Kepler	0	8	1	1	**10**
13.	Denisa Chytilová	GJŠ Přerov	8	0	1	0	**9**
	Kristýna Janovská	GMK Bílovec	0	8	0	1	**9**
15.	Yoram Baur	Graz Allstars	0	8	0	0	**8**
	Matthias Likawetz	Graz Allstars	0	8	0	0	**8**
	Kevin Nguyen	BRG Kepler	0	8	0	0	**8**
	Christian Thallinger	Graz Allstars	0	8	0	0	**8**
19.	Adrian Steinmann	BRG Kepler	0	3	0	1	**4**
20.	Anja Zotter	Graz Allstars	0	0	0	0	**0**

Category B (Individual Competition)

Rank	Name	School	1	2	3	4	∑
1.	Matouš Bílek	GJŠ Přerov	8	8	8	0	**24**
	Julian Narimany	AZSO Chorzów	8	8	8	0	**24**
	Clara Rinner	BRG Kepler	8	8	8	0	**24**
	Marcin Traskowski	AZSO Chorzów	8	8	8	0	**24**
5.	Thiemo Dsubanko	BRG Kepler	8	8	7	0	**23**
	Tomáš Křižák	GMK Bílovec	6	8	8	0	**22**
7.	Oskar Kleinoscheg	BRG Kepler	8	6	6	0	**20**
	Minh Nguyen	BRG Kepler	8	4	8	0	**20**
	Michaela Svatošová	GMK Bílovec	4	8	8	0	**20**
10.	Olaf Placha	AZSO Chorzów	4	7	8	0	**19**
11.	Teresa Reisner	Graz Allstars	0	8	8	0	**16**
12.	Jakub Michna	GMK Bílovec	0	7	8	0	**15**
	Vít Úlehla	GJŠ Přerov	0	7	8	0	**15**
14.	Jaroslav Hradil	GJŠ Přerov	0	4	8	0	**12**
15.	Matyáš Floríк	GMK Bílovec	3	4	1	0	**8**
	Alexandra Kahr	Graz Allstars+	0	8	0	0	**8**
17.	Sebastian Grosinger	Graz Allstars	6	0	1	0	**7**
18.	Melinda Bauer	Graz Allstars+	5	0	0	0	**5**
19.	Klara Balic	Graz Allstars	0	1	3	0	**4**
	Amálie Dostalíková	GJŠ Přerov	0	4	0	0	**4**
21.	Johannes Trattner	Graz Allstars+	0	1	0	0	**1**
22.	Lukas Novak	Graz Allstars	0	0	0	0	**0**
	Tomasz Szafarczyk	AZSO Chorzów	0	0	0	0	**0**

Category C (Individual Competition)

Rank	Name	School	1	2	3	4	$\sum$
1.	Anna Szymik	AZSO Chorzów	8	7	8	8	**31**
2.	Wojciech Socha	AZSO Chorzów	8	8	8	5	**29**
3.	Štěpán Postava	GMK Bílovec	5	6	8	6	**25**
4.	Flavio Arrigoni	BRG Kepler	8	1	7	8	**24**
5.	Matyáš Khýr	GMK Bílovec	8	0	8	7	**23**
6.	Lucie Dorazilová	GJŠ Přerov	6	1	8	6	**21**
	Darian Poljak	GJŠ Přerov	8	5	6	2	**21**
8.	Anna Krzywiecka	AZSO Chorzów	8	0	1	8	**17**
9.	Markus Auer	Graz Allstars	8	0	5	3	**16**
	Darius Buze	Graz Allstars	8	0	0	8	**16**
11.	Matteo Valentini	BRG Kepler	8	1	0	6	**15**
12.	Michal Bravanský	GMK Bílovec	8	1	0	4	**13**
13.	Sebastian Andritsch	BRG Kepler	5	0	3	4	**12**
14.	Inbal Rosenmann	BRG Kepler	3	0	0	8	**11**
15.	Marta Čermáková	GMK Bílovec	8	2	0	0	**10**
	David Tobolík	GJŠ Přerov	8	0	0	2	**10**
17.	Aleksandra Bedlicka	AZSO Chorzów	2	0	0	6	**8**
	Gabriel Nitsche	Graz Allstars	8	0	0	0	**8**
	Vendula Onderková	GJŠ Přerov	8	0	0	0	**8**
	Celina Rinner	Graz Allstars+	8	0	0	0	**8**
	Leon Wissounig	Graz Allstars	8	0	0	0	**8**
22.	Sarah Viertler	Graz Allstars+	0	0	0	5	**5**

Category A (Team Competition)

Rank	School	1	2	3	∑
1.	Chorzów	8	7	8	**23**
2.	Kepler	8	6	8	**22**
3.	Přerov	8	2	1	**11**
4.	Bílovec	4	3	2	**9**
5.	Allstars	1	0	1	**2**

Category B (Team Competition)

Rank	School	1	2	3	∑
1.	Kepler	8	0	8	**16**
	Allstars	8	0	8	**16**
3.	Přerov	4	0	8	**12**
4.	Chorzów	8	0	3	**11**
5.	Bílovec	3	0	6	**9**
6.	Allstars+	1	1	6	**8**

Category C (Team Competition)

Rank	School	1	2	3	∑
1.	Bílovec	8	8	7	**23**
2.	Chorzów	8	6	8	**22**
3.	Přerov	8	3	4	**15**
4.	Allstars	1	8	0	**9**
5.	Kepler	3	0	0	**3**
6.	Allstars+	0	0	0	**0**

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