XXV. Mathematical Duel - 2017

Descriptons of the solutions of the problems from inividual competetion category C

(solution by Štěpán Postava and Matyáš Khýr)

Problem #1

How many integers from the set {1, 2, . . . , 2017} are divisible by 11 but not by 7?

A description of the solution:

At first we have to determine how many integers from this set are divisible by 11.

$$2017 : 11 = 183 \ (residue: 4)$$

In this set there are 183 integers divisible by 11 included integeres divisible by 7 so we have to eliminate them.

$$2017 : 7 = 26 \ (residue: 15)$$

Now we know how many integeres are divisible by 11 and also by 7 wich we have to eliminate.

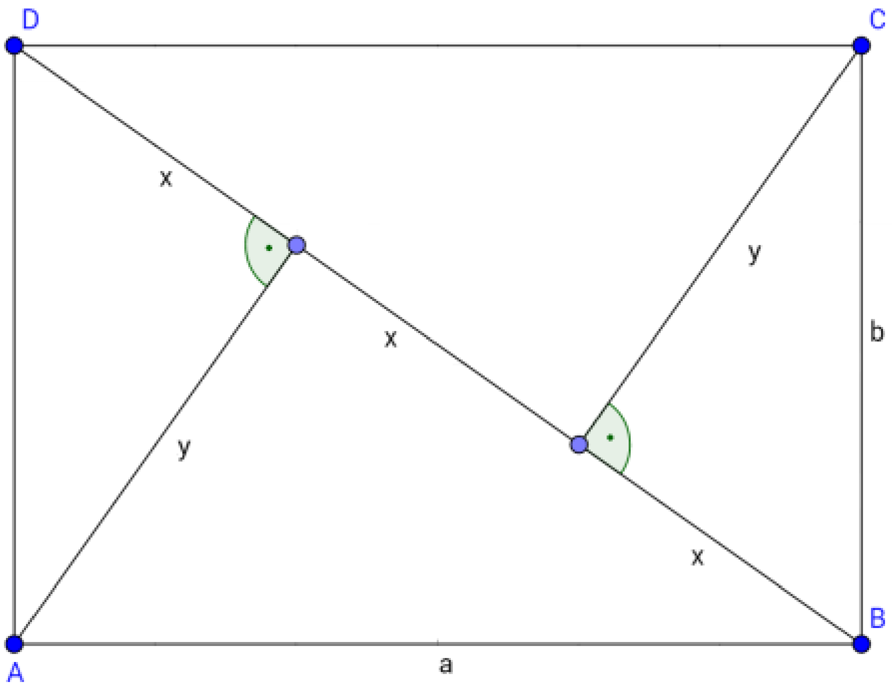
$$183 - 26 = 157$$

In the set {1, 2, . . . , 2017} are 157 integers divisible by 11 but not by 7.

Problem #2

We are given a rectangle *ABCD* with |*AB*|= *a* > *b* = |*BC*|. Let us suppose that the feet of the perpendiculars from the vertices A and C to diagonal BD divide this diagonal into three congruent segments.
Determine ratio *a* : *b*

A description of the solution:



$$y2 = a2 - 4x2$$
$$y2 = b2 - x2$$

$$a2 - 4x2 = b2 - x2$$
$$a2 - b2 = 3x2$$

$$a2 - b2 = 3x2$$
$$a2 + b2 = 9x2$$

$$a2 = 6x2$$
$$b2 = 3x2$$

$$\frac{a}{b} = \frac{\sqrt{6}x}{\sqrt{3}x} = \frac{\sqrt{2}}{1}$$

Ratio $a : b = \frac{\sqrt{2}}{1}$

Problem #3

We are given a system of equations

$$\boldsymbol{x + cy = c,}$$

$$\boldsymbol{2x + 4y = 3}$$

with unknowns *x* and *y*. Determine all possible values of a real parameter c such that equation $\boldsymbol{4x - y = 2}$ holds.

A description of the solution:

$$x + cy = c \;\Rightarrow x = c - cy$$

$$2x + 4y = 3 \;\Rightarrow \frac{3 - 4\mathrm{y}}{2}$$

$$c - cy = \frac{3 - 4\mathrm{y}}{2}$$

$$2c - 2cy = 3 - 4y$$

$$2x + 4y = 3 \;\Rightarrow x = \frac{3 - 4\mathrm{y}}{2}$$

$$4x - y = 2 \Rightarrow \; x = \frac{2 + \mathrm{y}}{\mathrm{y}}$$

$$\frac{3 - 4\mathrm{y}}{2} = \frac{2 + \mathrm{y}}{4}$$

$$6 - 8y = 2 + y$$

$$9y = 4$$

$$y = \frac{4}{9}$$

$$2c - 2c\times\frac{4}{9} = 3 - 4\times\frac{4}{9}$$

$$2c - \frac{8\mathrm{c}}{9} = 3 - \frac{16}{9}$$

$$18c - 8\,c = 27 - 16$$

$$10c = 11$$

$$c = \frac{11}{10}$$

Only one value of real parameter *c* is $\boldsymbol{\frac{11}{10}}$.

Problem #4

The Count of Lichenem likes to count, but he doesn't like most of the numbers. He likes a number if it has both even and odd digits, and he doesnt't like a number if it has even number of odd digits or an odd number of even digits.

a) How many numbers smaller than 100 does the Count like?
b) How many numbers smaller than 1 000 does the Count like?
c) How many numbers smaller than 10 000 does the Count like?

A description of the solution:

Numbers which the Count like have to be created by odd number of digits because the number of digits has to be created by odd number of odd digits and even number of even digits.

a) The Count of Lichenem doesn't like any number smaller than 100.

With three-digit numbers there may occur some possibilites because there are always two even and one odd digits:

If the first digit is even → the second one can be odd → the third one will has to be even

→ the second one can by even → the third one has to be odd

And if the first digit is odd → the second one has to be even → the third one has to be even

Every digit make some new possibilities of continuing in number.
We are choosing integers from the range ⟨0;9⟩.
But when the first digit is even, we can't use zero.

There are our possibilities again with numbers of variants:

The first digit is even (4) → the second one can be odd (5)→ the third one has to be even (5)

→ the second one can by even (5)→ the third one has to be odd (5)

If the first digit is odd (5) → the second one has to be even (5) → the third one has to be even (5)

$$4\times5\times5 + 4\times5\times5 + 5\times5\times5 = 325$$

b) The Count likes only 325 numbers smaller than 1000.

And because numbers smaller than 10 000 which have got odd number of digits, are exactly the same number as numbers smaller than 1 000 with odd number of digits there aren't any change in the third part of this problem.

c) The Count likes 325 numbers smaller than 10 000.