Problems on Numbers with Interesting Digits
some problems from the Mathematical Duel
Bílovec - Chorzów - Graz - Přerov

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The Mathematical Duel, categories

- A, B, C

The Mathematical Duel, categories

- A, B, C
- Individual, Team

C Team, Problem 2, 2013

We consider positive integers that are written in decimal notation using only one digit (possibly more than once), and call such numbers *uni-digit numbers*.

a) Determine a uni-digit number written with only the digit 7 that is divisible by 3.

b) Determine a uni-digit number written with only the digit 3 that is divisible by 7.

c) Determine a uni-digit number written with only the digit 5 that is divisible by 7.

d) Prove that there cannot exist a uni-digit number written with only the digit 7 that is divisible by 5.

C Team, Problem 2, 2013, Solution

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b) $333333 = 333 \cdot 1001 = 333 \cdot 7 \cdot 11 \cdot 13.$

c) $555555 = 555 \cdot 7 \cdot 11 \cdot 13.$

d) The last digit of any number divisible by 5 is always either 0 or 5. Any number that is divisible by 5 can therefore not be written using only the digit 7. $\square$

B Individual, Problem 4, 2013

We call a number that is written using only the digit 1 in decimal notation a *onesy* number, and a number using only the digit 7 in decimal notation a *sevensy* number. Determine a onesy number divisible by 7 and prove that for any sevensy number k, there always exists a onesy number m such that m is a multiple of k.

B Individual, Problem 4, 2013, Solution 1/2

onesy numbers: $111\ldots111$

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not divisible by 7: 1; 11; $111 = 3 \cdot 37$; $1111 = 11 \cdot 101$

B Individual, Problem 4, 2013, Solution 1/2

onesy numbers: $111\dots111$

not divisible by 7: 1; 11; $111 = 3 \cdot 37$; $1111 = 11 \cdot 101$

A possible onesy number divisible by seven is given by
$111111 = 111 \cdot 1001 = 111 \cdot 7 \cdot 11 \cdot 13.$ $\square$

sevensy number: $k = 777\ldots777$

B Individual, Problem 4, 2013, Solution 2/2

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In order to see that there always exists a onesy multiple of any sevensy number k, note that there exist an infinite number of onesy numbers. By the Dirichlet principle, there must therefore exist two different onesy numbers $m_1 > m_2$ with $m_1 \equiv m_2 \pmod{k}$.
$m_1 = 11111\dots111$
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$$m_1 - m_2 = 11\cdots1100\dots00$$

It therefore follows that $m_1 - m_2$ is divisible by k. The number $m_1 - m_2$ can be written as $m_1 - m_2 = m \cdot 10^r$, where m is also a onesy number. Since k is certainly not divible by 2 or 5, it follows that m must also be divisible by k, and the proof is complete. $\square$

B Team, Problem 1, 2005

a) A number x can be written using only the digit a both in base 8 and in base 16, i.e.

$$x = (aa\dots a)_8 = (aa\dots a)_{16}.$$

Determine all possible values of x.

b) Determine as many numbers x as possible that can be writen in the form $x = (11\dots 1)_b$ in at least two different number systems with bases b_1 and b_2. (*author unknown*) a). If $(aa\dots a)_8 = (aa\dots a)_{16}$ holds, there exist m and n such that

$$a \cdot 16^m + a \cdot 16^{m-1} + \cdots + a \cdot 16 + a = a \cdot 8^n + a \cdot 8^{n-1} + \cdots + a \cdot 8 + a$$

holds.

B Team, Problem 1, 2005, Solution 1/2

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holds. This is equivalent to

$$16^m + \cdots + 16 = 8^n + \cdots + 8 \iff 16\cdot\frac{16^m - 1}{16 - 1} = 8\cdot\frac{8^n - 1}{8 - 1}$$

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The right side is odd. Therefore, we have $n=0$, and $m=0$. The only possible values of a are $a\in\{0,1,2,\ldots,7\}$, and we therefore have $x\in\{0,1,2,\ldots,7\}$. $\square$

B Team, Problem 1, 2005, Solution 2/2

b) If $x = (11\dots1)_{b_1} = (11\dots1)_{b_2}$, we have $x = 1$ or $b_1, b_2 > 1$.

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Assume $1 < b_1 < b_2$. We want

$$x = \sum_{i=0}^{m} b_1^i = \sum_{j=0}^{n} b_2^j \quad \text{with} \quad m > n.$$

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Assume $1 < b_1 < b_2$. We want

$$x = \sum_{i=0}^{m} b_1^i = \sum_{j=0}^{n} b_2^j \quad \text{with} \quad m > n.$$

For any $b_1 > 1$ choose $b_2 = \sum_{i=1}^{m} b_1^i$. Then

$$(11)_{b_2} = 1 \cdot b_2 + 1 \cdot b_2^0 = \sum_{i=1}^{m} b_1^i + 1 \cdot b_1^0 = \sum_{i=0}^{m} 1 \cdot b_1^i = (11\dots1)_{b_1},$$

and we have infinitely many x with the required property. $\square$

A Team, Problem 3, 2013

We call positive integers that are written in decimal notation using only the digits 1 and 2 *Graz numbers*. Note that 2 is a 1-digit Graz number divisible by 2^1, 12 is a 2-digit Graz number divisible by 2^2 and 112 is a 3-digit Graz number divisible by 2^3.

a) Determine the smallest 4-digit Graz number divisible by 2^4.

b) Determine an n-digit Graz number divisible by 2^n for $n > 4$.

c) Prove that there must always exist an n-digit Graz number divisible by 2^n for any positive integer n.

A Team, Problem 3, 2013, Solution

some 4-digit candidates: 2222, 1212, 2112

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$$2222 = 2 \cdot 1111, \quad 1212 = 2^2 \cdot 303, \quad 2112 = 64 \cdot 33 = 2^6 \cdot 33$$

We can prove by induction that there in fact exists a unique n-digit Graz number for any positive integer n. Obviously, 2 is the only 1-digit Graz number, as 1 is not divisible by 2^1, but 2 is. We can therefore assume that there exists a unique k-digit Graz number g for some $k \geq 1$. Since g is divisible by 2^k, either $g \equiv 0 \pmod{2^{k+1}}$ or $g \equiv 2^k \pmod{2^{k+1}}$ must hold. Since $10^k \equiv 2^k \pmod{2^{k+1}}$ and $2 \cdot 10^k \equiv 0 \pmod{2^{k+1}}$, we have either $10^k + g \equiv 0 \pmod{2^{k+1}}$ or $2 \cdot 10^k + g \equiv 0 \pmod{2^{k+1}}$, and therefore the unique existence of an $n-1$-digit Graz number.
It is now easy to complete the solution. Since 112 is the 3-digit Graz number, and $112 = 16 \cdot 7$ is divisible by 16, 2112 is the 4-digit Graz number. Since $2112 = 32 \cdot 66$ is divisible by $2^5 = 32$, 22112 is the 5-digit Graz number, and the solution is complete. $\square$

C Individual, Problem 3, 2011

Determine the number of ten-digit numbers divisible by 4 which are written using only the digits 1 and 2. (*Józef Kalinowski*)

C Individual, Problem 3, 2011, Solution

for instance: $2212211112 = 4 \cdot 553052778$

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Any ten-digit number n divisible by 4 must end in a two-digit number divisible by 4. The last two digits of any such number written only with the digits 1 and 2 can therefore only be 12, in this order. Each of the eight other eight digits of the ten-digit number can be either 1 or 2. Altogether, this gives us 2^8 possibilities.

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There therefore exist $2^8 = 256$ ten-digit numbers with the given property. $\square$

B Individual, Problem 1, 2011

Let A be a six-digit positive integer which is formed using only the two digits x and y. Furthemore, let B be the six-digit integer resulting from A if all digits x are replaced by y and simultaneously all digits y are replaced by x. Prove that the sum $A + B$ is divisible by 91. (*Józef Kalinowski*) for example: $229299 + 992922 = 1222221 = 91 \cdot 13431$

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Let $A = \overline{c_5c_4c_3c_2c_1c_0}$ and $B = \overline{d_5d_4d_3d_2d_1d_0}$, where $c_i, d_i \in \{x, y\}$, $c_i \neq d_i$ for $i = 0, 1, 2, 3, 4, 5$ and $x, y \in \{1, \ldots, 9\}$ are distinct non-zero decimal digits.
Since $c_i + d_i = x + y \neq 0$ for $i = 0, 1, 2, 3, 4, 5$ we have

$$A + B = (x + y) \cdot (10^5 + 10^4 + 10^3 + 10^2 + 10 + 1) =$$

$$= (x + y) \cdot 111111 = (x + y) \cdot 91 \cdot 1221,$$

The number $A + B$ is therefore certainly divisible by 91. $\square$

C Team, Problem 1, 2010

Determine the number of pairs (x, y) of decimal digits such that the positive integer in the form $\overline{xyx}$ is divisible by 3 and the positive integer in the form $\overline{yxy}$ is divisible by 4. (*author unknown*)

C Team, Problem 1, 2010, Solution

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Each positive integer in the form $\overline{yxy}$ is divisible by 4 if and only if the number $\overline{xy}$ is divisible by 4 with $y \neq 0$. Hence

$$(x, y) \in \{(1;2), (1;6), (2;4), (2;8), (3;2), (3;6), (4;4), (4;8), (5;2), \ldots$$

$$\ldots, (5;6), (6;4), (6;8), (7;2), (7;6), (8;4), (8;8), (9;2), (9;6)\}.$$

A positive integer in the form $\overline{xyx}$ is divisible by 3 if and only if the sum of its digits is divisible by 3, i.e. $2x + y$ must be divisible by 3. After checking all possible pairs of positive integers we obtain only six possibilities:

$$(x, y) \in \{(2;8), (3;6), (4;4), (5;2), (8;8), (9;6)\}.$$

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$$(x, y) \in \{(2;8), (3;6), (4;4), (5;2), (8;8), (9;6)\}.$$

We therefore have 6 solutions altogether. $\square$

B Team, Problem 3, 2015

Determine the number of all six-digit palindromes that are divisible by seven.
[*Remark.* A six-digit palindrome is a positive integer written in the form $\overline{abccba}$ with decimal digits $a \neq 0$, b and c.] (*Pavel Calábek*)

B Team, Problem 3, 2015, Solution

We have

$$\overline{abccba} = 100001a + 10010b + 1100c$$
$$= 7(14286a + 1430b + 157c) - (a - c).$$

Such a number is divisible by 7 iff $(a - c)$ is divisible by 7.

B Team, Problem 3, 2015, Solution

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Such a number is divisible by 7 iff $(a - c)$ is divisible by 7. $a \neq 0$ and c are decimal digits, and we therefore have $-8 \leq a - c \leq 9$. The only possible values are therefore $(a - c) \in \{-7, 0, 7\}$.

B Team, Problem 3, 2015, Solution

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For $a - c = -7$ we have $(a, c) \in \{(1, 8), (2, 9)\}$,
for $a - c = 0$ we have $(a, c) \in \{(1, 1), (2, 2), \ldots, (9, 9)\}$
and finally for $a - c = 7$ we have $(a, c) \in \{(7, 0), (8, 1), (9, 2)\}$,
and therefore 14 possibilities for the ordered pair (a, c).

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and therefore 14 possibilities for the ordered pair (a, c).
In all cases b is an arbitrary digit, and altogether there are therefore
$14 \cdot 10 = 140$ six-digit palindromes which are divisible by 7. $\square$

C Individual, Problem 3, 2012

Two positive integers are called *friends* if each is composed of the same number of digits, the digits in one are in increasing order and the digits in the other are in decreasing order, and the two numbers have no digits in common (like for example the numbers 147 and 952).
Solve the following problems:
a) Determine the number of all two-digit numbers that have a friend.
b) Determine the largest number that has a friend.

a) Every two-digit number n which is composed of different digits, has its digits in increasing or decreasing order. Moreover there are at least two non-zero digits a and b different from the digits of n. It follows, that the friend of n is one of numbers $\overline{ab}$ or $\overline{ba}$.

a) Every two-digit number n which is composed of different digits, has its digits in increasing or decreasing order. Moreover there are at least two non-zero digits a and b different from the digits of n. It follows, that the friend of n is one of numbers $\overline{ab}$ or $\overline{ba}$.
The number of two-digit numbers with a friend is therefore equal to the number of two-digit numbers composed of different digits. There are 90 two-digit numbers of which 9 $(11, 22, \ldots, 99)$ consist of two identical digits. There are therefore 81 two-digit numbers which have a friend. $\square$

b) If the number with a friend has k digits, its friend also has k different digits and together they have $2k$ different digits. Since there are 10 digits, the largest number with a friend has at most 5 digits.

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No number begins with 0, so 0 is in a number with digits in decreasing order if $k = 5$. Moreover, if a number n with digits in increasing order has a friend k, its mirror image (that is, the number with the same digits in opposite order) is greater and has a friend (namely the mirror image of k).

b) If the number with a friend has k digits, its friend also has k different digits and together they have $2k$ different digits. Since there are 10 digits, the largest number with a friend has at most 5 digits.
No number begins with 0, so 0 is in a number with digits in decreasing order if $k = 5$. Moreover, if a number n with digits in increasing order has a friend k, its mirror image (that is, the number with the same digits in opposite order) is greater and has a friend (namely the mirror image of k).
The largest number with a friend has different digits in decreasing order, it has at most five digits and one of its digits is 0. The largest such number is therefore 98760 and its friend is 12345. $\square$

C Team, Problem 3, 2015

A *wavy number* is a number in which the digits alternately get larger and smaller (or smaller and larger) when read from left to right. (For instance, 3629263 and 84759 are wavy numbers but 45632 is not.)

a) Two five-digit wavy numbers m and n are composed of all digits from 0 to 9. (Note that the first digit of a number cannot be 0.) Determine the smallest possible value of $m + n$.

b) Determine the largest possible wavy number in which no digit occurs twice.

c) Determine a five-digit wavy number that can be expressed in the form $ab + c$, where a, b and c are all three-digit wavy numbers.

C Team, Problem 3, 2015, Solution

a) The smallest possible sum is given by the expression

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c) There are many such combinations. Examples are

$$120 \cdot 142 + 231 = 17271 \quad \text{or} \quad 101 \cdot 101 + 101 = 10302.$$

Thank you!

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